Filtered Deformations of Trivariate Polynomial Rings over a Field

by

Boris Li

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Author's Declaration

I hereby declare that I am the sole author of this research paper. This is a true copy of the research paper, including any required final revisions, as accepted by my examiners.

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Abstract

We demonstrate that all filtered deformations of trivariate polynomial rings over an algebraically closed field of positive characteristic satisfy a polynomial identity. We do so by first classifying up to isomorphism all possible deformations over a field of any characteristic and not necessarily algebraically closed, describing them either as iterated Ore extensions or associative algebras generated by three variables that satisfy a set of commutators. This affirms a question of Etingof in a special case.

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Dedication

For a struggling student in his first abstract algebra class: look at how far you have come.

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Chapter 1

Introduction

As non-commutativity naturally arises from quantum mechanics, in the field of quantum algebras, we are interested in studying non-commutative, associative algebras. Of particular interest are deformations, which allow us to tweak commutative algebras and introduce some element of non-commutativity into the multiplicative structure of the algebra.

In the case of enveloping algebras in positive characteristic, it is well-known that they satisfy a polynomial identity. A natural line of inquiry is to ask whether this phenomenon extends more generally. In [8, Question 1.1], Etingof poses the following question: if K is a field of characteristic $p > 0$, and A is a filtered deformation of a commutative finitely generated domain A_0 over K, does A necessarily satisfy a polynomial identity? This is also related to the question posed in the introduction of [7] by Cuadra, Etingof, and Walton, and would stand to be true if Etingof's question has an affirmative answer.

This question remains unanswered to this day. However, Brown and Zhang [6] have shown that iterated Hopf Ore extensions do satisfy this property in positive characteristic. More generally, when A_0 is finitely generated over K, Small and Warfield gave an affirmative answer in [11, 12] for the case of A_0 being Krull dimension 1, and Bell later affirmed the question in [4] for the case of Krull dimension 2. The next natural cases to consider are thus polynomial rings in 3 variables, which have Krull dimension 3.

In this work, we shall answer Etingof's question positively for the case of A_0 being a trivariate polynomial ring $K[x, y, z]$.

Theorem 1.1. *Suppose K is an algebraically closed field of characteristic $p > 0$. Any filtered deformation of $K[x, y, z]$ is a polynomial identity ring.*

In the process of doing so, we explicitly classify all such filtered deformations up to isomorphism.

Theorem 1.2. *Suppose K is a field. A filtered deformation of $K[x, y, z]$ must be isomorphic to either one of two types of iterated Ore extensions:*

(I) $K[x][y, \varepsilon][z; \sigma, \delta]$, *or*
(II) $K[x][y; \tau][z; \sigma, \delta]$,

or an associative algebra governed by any one of the following possible sets of commutators:

(IIIac) when K has characteristic not 2,

$$\begin{aligned}
[x, y] &= z, \\
[x, z] &= \alpha z + y^2 - 2\alpha xy + (\alpha x^3 + \beta x^2 + \gamma x + \delta), \quad \textit{and} \\
[y, z] &= (3\alpha x + \beta - \alpha^2)z - (3\alpha x^2 + 2(\beta - \alpha^2)x + \gamma)y + f(x)
\end{aligned}$$

for $\alpha, \beta, \gamma, \delta \in K$ and some polynomial $f(x) \in K[x]$ of degree at most 4;

(IIIb) when K has characteristic neither 2 nor 3,

$$[x, y] = z, \quad [x, z] = \alpha z + y^2 + \delta, \quad \textit{and} \quad [y, z] = 0$$

for $\alpha, \delta \in K$, where $\alpha \neq 0$;

(IIId) when K has characteristic 3,

$$\begin{aligned}
[x, y] &= z, \\
[x, z] &= \beta z + y^2 + \alpha xy + (\alpha x^3 + \alpha^2 x^2 - \alpha(\alpha - \beta)^2 x + \delta), \quad \textit{and} \\
[y, z] &= \alpha(\alpha - \beta)z
\end{aligned}$$

for $\alpha, \beta, \delta \in K$, where $\alpha \neq \beta$;

(IIIe) when K has characteristic 2,

$$\begin{aligned}
[x, y] &= z, \\
[x, z] &= \alpha z + y^2 + \varepsilon y + (\alpha x^3 + \beta x^2 + \gamma x + \delta), \quad \textit{and} \\
[y, z] &= (\alpha x + \alpha^2 + \beta)z + (\alpha x^2 + \alpha\varepsilon + \gamma)y + f(x)
\end{aligned}$$

for $\alpha, \beta, \gamma, \delta, \varepsilon \in K$ and some polynomial $f(x) \in K[x]$ of degree at most 4;

(IIIf) *when K has characteristic 2,*

$$[x, y] = z, \quad [x, z] = \alpha z + y^2 + \delta y + (\beta x^2 + \alpha\beta x + \gamma), \quad \textit{and} \quad [y, z] = \beta z$$

for $\alpha, \beta, \gamma, \delta \in K$, where $\alpha \neq 0$;

(IVa) *when K has characteristic not 2,*

$$[x, y] = z, \quad [x, z] = z - 2xy + \delta, \quad \textit{and} \quad [y, z] = y^2 + f(x)$$

for $\delta \in K$ and some arbitrary polynomial $f(x) \in K[x]$;

(IVbi) *regardless of the characteristic of K,*

$$[x, y] = z, \quad [x, z] = z + \alpha y, \quad \textit{and} \quad [y, z] = 0$$

for nonzero $\alpha \in K$;

(IVbii) *when K has characteristic 2,*

$$[x, y] = z, \quad [x, z] = z + \alpha y, \quad \textit{and} \quad [y, z] = y^2 + f(x)$$

for nonzero $\alpha \in K$ and some arbitrary $f(x) \in K[x]$;

(IVc) *regardless of the characteristic of K,*

$$[x, y] = z, \quad [x, z] = \alpha y, \quad \textit{and} \quad [y, z] = f(x)$$

for nonzero $\alpha \in K$ and some arbitrary $f(x) \in K[x]$;

(IVdi) *when K has characteristic $p > 0$,*

$$[x, y] = z, \quad [x, z] = z + g(x^p - x), \quad \textit{and} \quad [y, z] = f(x)z + (f(x) + g^{(1)}(x^p - x))g(x^p - x)$$

for arbitrary polynomials $f(x), g(x) \in K[x]$;

(IVdii) *when K has characteristic 0,*

$$[x, y] = [x, z] = z \quad \textit{and} \quad [y, z] = f(x)z$$

for some arbitrary polynomial $f(x) \in K[x]$;

(IVdiii) *when K has characteristic 2,*

$$[x, y] = [x, z] = z \quad \textit{and} \quad [y, z] = f(x)z + y^2 + f(x)y + g(x)$$

for arbitrary polynomials $f(x), g(x) \in K[x]$;

(IVdiv) when K has characteristic 2,

$$\begin{aligned}
&[x, y] = z, \\
&[x, z] = z + f(x), \quad \textit{and} \\
&[y, z] = (f(x+1) + f(x) + f^{(1)}(x))z + y^2 + f^{(1)}(x)y + g(x)
\end{aligned}$$

for arbitrary polynomials $f(x), g(x) \in K[x]$;

(IVei) regardless of the characteristic of K,

$$[x, y] = z, \quad [x, z] = 0, \quad \textit{and} \quad [y, z] = f(x)y + g(x)$$

for arbitrary polynomials $f(x), g(x) \in K[x]$; or

(IVeii) regardless of the characteristic of K,

$$[x, y] = z, \quad [x, z] = f(x), \quad \textit{and} \quad [y, z] = f^{(2)}(x)z - f^{(1)}(x)y + g(x)$$

for arbitrary polynomials $f(x), g(x) \in K[x]$.

This paper will be organized as such: we shall spend Chapter 2 exhaustively showing all the possible algebras that our deformation can be, and then quickly showing each algebra is indeed a deformation. We then use this classification in Chapter 3 to show that each case indeed satisfies a polynomial identity.

Chapter 2

Classification

In this chapter, we shall prove Theorem 1.2 by classifying all filtered deformations of $K[x, y, z]$, where K is a field, and x, y, z our three generators.

We shall assume, for the remainder of this chapter, that K is a fixed field, and let A be a filtered deformation of $K[x, y, z]$.

Definition 2.1 (Filtered deformation)**.** Recall that an *ascending filtration*[1] of a K-algebra A is a sequence of subspaces

$$K = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq \bigcup_{n=0}^{\infty} V_n = A$$

obeying $V_i V_j \subseteq V_{i+j}$ for all $i, j \geq 0$. Here, in particular, each V_n is a finite-dimensional K-vector space.

We say A is a *filtered deformation* of another K-algebra A_0 if the *associated graded algebra* $V_0 \oplus \bigoplus_{n=1}^{\infty} V_n / V_{n-1}$ of A with respect to this filtration is isomorphic to A_0.

Example 2.2 (Weyl algebra)**.** Let A_1 be the ring of elements consisting of $K[x]$-linear combinations of powers of the partial differential operator ∂_x; that is, A_1 contains elements of the form $\sum_{i=0}^{\infty} f_i(x) \partial_x^i$, where almost all the $f_i \in K[x]$ are zero. This is equipped with the commutator $[\partial_x, f(x)] = \frac{\partial}{\partial x} f(x)$ that evaluates the partial derivative. We call A_1 the first Weyl algebra, which can also be expressed as the quotient of a free algebra with two generators x and ∂_x, by the ideal generated by $[\partial_x, x] - 1$.

Observe that if we let x and ∂_x both belong in V_1, then $\operatorname{gr} A_1$ is simply a bivariate polynomial ring over K, so we say that A_1 is a filtered deformation of $K[x, y]$.

[1] All filtrations from here onwards are ascending and graded by $\mathbb{Z}_{\geq 0}$.

Given some nonzero element $u \in A$, we let $\deg(u)$ denote the smallest nonnegative integer n such that $u \in V_n$, and call this the degree of u. In particular, we shall let the degree of 0 to be $-\infty$. Because the associated graded ring of A is commutative, we have the inequality

$$\deg([u, v]) \leq \deg(u) + \deg(v) - 1$$

for $u, v \in A$. Moreover, since the associated graded ring of A is a polynomial algebra in three variables, A has generators x, y, z, and has a K-basis given by monomials $x^i y^j z^k$ where $i, j, k \geq 0$, as observed in [5]. Allow us to denote by a, b, c the degrees of x, y, z respectively, and without any loss of generality, assume $a \leq b \leq c$. Notice that $x^i y^j z^k$ has degree $ai + bj + ck$.

As $\deg([x, y]) \leq a + b - 1$, $[x, y]$ must be a linear combination of monomials $x^i y^j z^k$ with either:

(i) $k = 1$ and $i = j = 0$;

(ii) $j = 1$ and $i = k = 0$; or

(iii) $j = k = 0$.

In particular, if the coefficient of z is nonzero, then we have $[x, y] = \lambda z + u(x, y)$ for some nonzero $\lambda \in K$ and polynomial $u \in K[x, y]$. Substituting $\lambda z + u(x, y)$ for our generator z, we may assume, without any loss of generality, that $[x, y] = z$.

On the other hand, if z does not appear in our expression for $[x, y]$, then we have $[x, y] = \lambda y + u(x)$ for some $u \in K[x]$. If λ is nonzero, we may substitute $\lambda^{-1} x$ and $\lambda y + u(x)$ for our generators x and y respectively, and thus, without any loss of generality, assume $[x, y] = y$.

Therefore, it suffices to assume that our algebra A has three generators x, y, z with degrees $a \leq b \leq c$ respectively, with $[x, y]$ appearing as one of the following three forms:

(I) $[x, y] = u(x)$,

(II) $[x, y] = y$, or

(III) $[x, y] = z$.

It turns out that the last case is the most difficult, while the other two cases are considerably simpler. We thus classify filtered deformations in Cases I and II first, and then further divide the last case into two:

(III) $[x, y] = z$, and the coefficient of y^2 in $[x, z]$ is nonzero; and

(IV) $[x, y] = z$, and the coefficient of y^2 in $[x, z]$ vanishes.

2.1 Preliminaries

Before we dedicate our time to studying each of the possible cases, let us first introduce a few important definitions, and prove a few tools that will later be useful.

Definition 2.3 (Skew polynomial ring)**.** Suppose R is a ring, not necessarily commutative. Let $\sigma\colon R \to R$ be a ring homomorphism, and let $\delta\colon R \to R$ be a σ-derivation of R, i.e. δ is an additive (abelian) group homomorphism satisfying

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b.$$

An *Ore extension of*, or alternatively, a *skew polynomial ring over* R, is the ring $R[x;\sigma,\delta]$, which is the set $R[x]$ equipped with the identity

$$xr = \sigma(r)x + \delta(r).$$

When δ is the zero map, the Ore extension is denoted $R[x;\sigma]$; when σ is the identity map, then the Ore extension is also called a differential polynomial ring, and is denoted $R[x,\delta]$.

Example 2.4 (Weyl algebra)**.** The first Weyl algebra A_1 is the differential polynomial ring $K[x][\partial_x, \frac{\partial}{\partial x}]$.

Similarly, we can define the nth Weyl algebra A_n as the K-algebra with $2n$ generators $\{x_1,\dots,x_n,\partial_1,\dots,\partial_n\}$, governed by the commutators

$$[x_i,x_j] = [\partial_i,\partial_j] = 0 \quad \text{and} \quad [\partial_i,x_j] = \delta_{ij} \quad \text{for} \quad i,j \in \{1,\dots,n\},$$

where δ_{ij} is the Kronecker delta. Then A_n is the iterated Ore extension

$$K[x_1,\dots,x_n]\left[\partial_1, \frac{\partial}{\partial x_1}\right]\cdots\left[\partial_n, \frac{\partial}{\partial x_n}\right],$$

where we force the partial derivatives to commute.

Since we will be working not only in characteristic 0, but also characteristic $p > 0$, we occasionally encounter the problem of taking derivatives over monomials such as x^p, which evaluate to zero even though our monomial is not constant. We remedy this by introducing a different derivative that is based on the binomial coefficient, which is well-defined even in positive characteristic.

Definition 2.5 (Hasse-Schmidt derivative)**.** Suppose $f \in K[x]$ is a univariate polynomial. Let $f^{(n)}$ denote its *nth Hasse-Schmidt derivative*, for integers $n \geq 0$. If f is a monomial x^m, it shall obey, for $n \geq 1$:

(i) $f^{(0)}(x) = x^m$;

(ii) $f^{(n)}(x) = \binom{m}{n} x^{m-n}$ if $m \geq n$; and

(iii) $f^{(n)}(x) = 0$ if $m < n$.

This is then extended K-linearly to polynomials.

Observe that in characteristic 0, we have the equality

$$f^{(n)}(x) = \frac{1}{n!} \frac{d^n}{dx^n} f(x).$$

We digress, and prove two tools that will later become useful in our exhaustive proof. The first tool is a formula for the commutator $[f(x), y]$, involving a univariate polynomial f, and non-commutative elements x, y. This is likely a well-known and classical result, but we are unable to find a source.

Proposition 2.6 (Polynomial commutator)**.** *Let K be a field. Suppose x, y are arbitrary elements in a (non-commutative) K-algebra, and $f \in K[t]$ is a polynomial. Then the adjoint operator* $\mathrm{ad}_y = [y, -]$ *applied to a univariate polynomial evaluates to*[2]

$$\mathrm{ad}_y(f(x)) = \sum_{n=1}^{\infty} (-1)^n f^{(n)}(x)\, \mathrm{ad}_x^n(y).$$

Proof. It suffices to prove this for a monomial x^m, since the adjoint operator and the nth Hasse-Schmidt derivative are both linear operators. For $m = 0$, $x^0 = 1$, so $\mathrm{ad}_y(1) = 0$ can be written as an empty sum.

Allow us to proceed by induction. Assuming this formula holds for x^m, let us prove the

[2]Note that our sum is actually finite, because only finitely many derivatives of f are nonzero.

result for x^{m+1}. We simply compute:

$$\begin{aligned}
\operatorname{ad}_y(x^{m+1}) = \operatorname{ad}_y(x^m)x + x^m \operatorname{ad}_y(x) &= \sum_{n=1}^{\infty} (-1)^n \binom{m}{n} x^{m-n} \operatorname{ad}_x^n(y)x - x^m \operatorname{ad}_x(y) \\
&= \sum_{n=1}^{\infty} (-1)^n \binom{m}{n} x^{m-n} \big(x \operatorname{ad}_x^n(y) - \operatorname{ad}_x^{n+1}(y)\big) - x^m \operatorname{ad}_x(y) \\
&= \sum_{n=1}^{\infty} (-1)^n \binom{m}{n} x^{m-n+1} \operatorname{ad}_x^n(y) + \sum_{n=0}^{\infty} (-1)^{n+1} \binom{m}{n} x^{m-n} \operatorname{ad}_x^{n+1}(y) \\
&= \sum_{n=1}^{\infty} (-1)^n \binom{m}{n} x^{m-n+1} \operatorname{ad}_x^n(y) + \sum_{n=1}^{\infty} (-1)^n \binom{m}{n-1} x^{m-n+1} \operatorname{ad}_x^n(y) \\
&= \sum_{n=1}^{\infty} (-1)^n \binom{m+1}{n} x^{m-n+1} \operatorname{ad}_x^n(y),
\end{aligned}$$

which proves our result. □

We now state a fact about periodic polynomials in positive characteristic, which should be a classical result in Artin-Schreier theory. Nevertheless, as we cannot find a proof of this, we provide the following short proof.

Proposition 2.7. *Suppose K is an infinite field of characteristic $p > 0$. Then a polynomial $f \in K[x]$ that satisfies $f(x+1) = f(x)$ for all $x \in K$ must be a polynomial of $x^p - x$; that is, $f(x) = \tilde{f}(x^p - x)$ for some other polynomial $\tilde{f}$.*

Proof. If f is constant, then we are immediately done.

Now allow us to induct on the degree of polynomial f. Observe that we have $f(0) = f(1) = \cdots = f(p-1)$. Let $g(x) = f(x) - f(0)$, so that $\{0, 1, \ldots, p-1\}$ are all roots of the polynomial g. Thus, we know that $x(x-1)\cdots(x-p+1) = x^p - x$ is a factor of $g(x)$; that is, $f(x) = (x^p - x)q(x) + f(0)$. But then $q(x)$ is a polynomial of degree less than that of $f(x)$, but still obeys $q(x) = q(x+1)$, so the inductive hypothesis applies, and we are done. □

Notice that the inductive step does not work when we are working over a finite field $\mathbb{F}_{p^n}$, because once you factor out $x^{p^n} - x$, the remaining polynomial does not have any more constraints on it. However, if we work over the algebraic closure of such a field, we obtain an infinite field of the same characteristic, so the result holds there.

2.2 Case I: $[x, y] = u(x)$

First consider our algebra A with the constraint $[x, y] = u(x)$. We immediately observe that the subalgebra B of A generated by x, y must be a differential polynomial ring $K[x][y, \varepsilon]$ where $\varepsilon(x) = -u(x)$.

Now notice that the monomial $x^i y^j z^k$ has degree greater than $a + c - 1$ when $k \geq 1$ and $i + j \geq 1$, so we know that $[x, z]$ must be a linear combination of z and $x^i y^j$ only, meaning that $[x, z] = \alpha z + f(x, y)$ for some $\alpha \in K$ and $f \in B$. Similarly, $[y, z]$ must be a linear combination of $x^i z$ and $x^i y^j$ only, so we can write $[y, z] = h(x)z + g(x, y)$ for some $h \in K[x]$ and $g \in B$.

The Jacobi identity thus evaluates to

$$\begin{aligned}
[[x, y], z] &= [[x, z], y] + [x, [y, z]], \\
[u(x), z] &= -\alpha[y, z] + [f(x, y), y] + h(x)[x, z] + [x, g(x, y)], \quad \text{and} \\
[u(x), z] &= -\alpha g(x, y) + h(x)f(x, y) + [f(x, y), y] + [x, g(x, y)]. \qquad (2.1)
\end{aligned}$$

Notice that the right side of this equation does not contain z, so we can conclude that the coefficient of z in $\mathrm{ad}_z(u(x))$ must vanish.

To compute $\mathrm{ad}_z(u(x))$, we must first compute $\mathrm{ad}_x^n(z)$ for integers $n \geq 0$, which we claim to be

$$\mathrm{ad}_x^n(z) = \alpha^n z + \sum_{j=0}^{n-1} \alpha^{n-1-j} \,\mathrm{ad}_x^j f(x, y), \qquad (2.2)$$

and obviously this holds for $n = 0$. Proceeding inductively, if we assume the result for n, we can compute that

$$\begin{aligned}
\mathrm{ad}_x^{n+1}(z) &= \alpha^n \,\mathrm{ad}_x(z) + \sum_{j=0}^{n-1} \alpha^{n-1-j} \,\mathrm{ad}_x^{j+1} f(x, y) \\
&= \alpha^{n+1} z + \alpha^n f(x, y) + \sum_{j=1}^{n} \alpha^{n-j} \,\mathrm{ad}_x^j f(x, y) \\
&= \alpha^{n+1} z + \sum_{j=0}^{n} \alpha^{n-j} \,\mathrm{ad}_x^j f(x, y)
\end{aligned}$$

which indeed satisfies Equation 2.2. Thus, the polynomial commutator equation (Proposition 2.6) gives us

$$\operatorname{ad}_z(u(x)) = \sum_{n=1}^{\infty} (-1)^n u^{(n)}(x) \left(\alpha^n z + \sum_{j=0}^{n-1} \alpha^{n-1-j} \operatorname{ad}_x^j f(x,y) \right),$$

and the coefficient of z, which should vanish, is

$$0 = \sum_{n=1} (-\alpha)^n u^{(n)}(x) = u(x-\alpha) - u(x).$$

This gives us a constraint on the behaviour of u when $\alpha \neq 0$.

We now claim that A is a skew polynomial ring over B.

Lemma 2.8. *Let σ and δ be a k-algebra homomorphism and a σ-derivation respectively, given by*

$$\begin{array}{rccl} \sigma: & B & \to & B \\ & x & \mapsto & x-\alpha \\ & y & \mapsto & y-h(x) \end{array} \qquad \textit{and} \qquad \begin{array}{rccl} \delta: & B & \to & B \\ & x & \mapsto & -f(x,y) \\ & y & \mapsto & -g(x,y). \end{array}$$

Then our algebra R in this case is isomorphic to $A[z;\sigma,\delta]$.

Proof. We first verify that σ is indeed a k-algebra homomorphism.

$$\begin{aligned} \sigma([x,y]) &= \sigma(x)\sigma(y) - \sigma(y)\sigma(x) = (x-\alpha)(y-h(x)) - (y-h(x))(x-\alpha) \\ &= xy - yx = u(x) = u(x-\alpha) = \sigma(u(x)) \end{aligned}$$

We then verify that δ is indeed a σ-derivation.

$$\begin{aligned} \delta([x,y]) &= \delta(xy) - \delta(yx) = \sigma(x)\delta(y) + \delta(x)y - \sigma(y)\delta(x) - \delta(y)x \\ &= -(x-\alpha)g(x,y) - f(x,y)y + (y-h(x))f(x,y) + g(x,y)x \\ &= -[x,g(x,y)] - [f(x,y),y] + \alpha g(x,y) - h(x)f(x,y) = -[u(x),z] \end{aligned}$$

For this part, it remains to show that $\delta(u(x)) = \operatorname{ad}_z(u(x))$. We claim that

$$\delta(x^n) = \operatorname{ad}_z(x^n)\Big|_{z=0}$$

for any integer $n \geq 0$. Using this claim, we would know that $\delta(u(x)) = \operatorname{ad}_z(u(x))\big|_{z=0}$, and since we know from Equation 2.1 that z does not occur in $\operatorname{ad}_z(u(x))$, we would have

$$\delta(u(x)) = \operatorname{ad}_z(u(x))\Big|_{z=0} = \operatorname{ad}_z(u(x))$$

as desired.

We now prove our claim. Obviously, for $n = 0$ and 1, we have $\delta(1) = \mathrm{ad}_z(1) = 0$ and $\delta(x) - \mathrm{ad}_z(x) = \alpha z$. Proceeding by induction, suppose we have $\delta(x^n) = \mathrm{ad}_z(x^n)\big|_{z=0}$. We can calculate that

$$\begin{aligned}\delta(x^{n+1}) - \mathrm{ad}_z(x^{n+1}) &= (x-\alpha)\delta(x^n) + \delta(x)x^n - x\,\mathrm{ad}_z(x^n) - \mathrm{ad}_z(x)x^n \\ &= x(\delta(x^n) - \mathrm{ad}_z(x^n)) + (\delta(x) - \mathrm{ad}_z(x))x^n - \alpha\delta(x^n) \\ &= x(\delta(x^n) - \mathrm{ad}_z(x^n)) + \alpha z x^n - \alpha\delta(x^n) \\ &= x(\delta(x^n) - \mathrm{ad}_z(x^n)) + \alpha x^n z + \alpha(\mathrm{ad}_z(x^n) - \delta(x^n)) \\ &= (x-\alpha)(\delta(x^n) - \mathrm{ad}_z(x^n)) + \alpha x^n z.\end{aligned}$$

Evaluating at $z = 0$, the first term vanishes by the inductive hypothesis, and the second term simply vanishes as desired.

Finally, we now show that the other two commutators give an equivalent criterion to the Ore extension identity.

$$\begin{aligned}zx &= xz - [x,z] = xz - \alpha z - f(x,y) = (x-\alpha)z - f(x,y) = \sigma(x)z + \delta(x) \\ zy &= yz - [y,z] = yz - h(x)z - g(x,y) = (y - h(x))z - g(x,y) = \sigma(y)z + \delta(y)\end{aligned}$$

Therefore, our algebra $R = A[z;\sigma,\delta]$. □

Henceforth, $R = k[x][y,\varepsilon][z;\sigma,\delta]$, which is an iterated Ore extension.

2.3 Case II: $[x,y] = y$

Then consider our algebra A with the constraint $[x,y] = y$. We can again immediately observe that the subalgebra B of A generated by x, y must be a skew polynomial ring $K[x][y;\tau]$ where $\tau(x) = x - 1$.

As in our previous case, we have

$$[x,z] = \alpha z + f(x,y) \quad \text{and} \quad [y,z] = h(x)z + g(x,y)$$

for some $\alpha \in K$, $f, g \in B$, and $h \in K[x]$.

The Jacobi identity thus evaluates to

$$\begin{aligned}
[[x, y], z] &= [[x, z], y] + [x, [y, z]] \\
[y, z] &= [\alpha z, y] + [p(x, y), y] + [x, r(x)z] + [x, q(x, y)] \\
(1 + \alpha)(r(x)z + q(x, y)) &= r(x)[x, z] + [p(x, y), y] + [x, q(x, y)] \\
r(x)z &= r(x)p(x, y) - (1 + \alpha)q(x, y) + [p(x, y), y] + [x, q(x, y)]. \quad (2.3)
\end{aligned}$$

Again, notice that z does not occur at all on the right side of the equation, so we can conclude that $r(x) = 0$. It therefore suffices to analyze the algebra governed by

$$[x, y] = y, \quad [x, z] = \alpha z + f(x, y), \quad \text{and} \quad [y, z] = g(x, y). \quad (2.4)$$

We now claim that A is a skew polynomial ring over B.

Lemma 2.9. *Let σ and δ be a k-algebra homomorphism and a σ-derivation respectively, given by*

$$\begin{array}{rlcl}
\sigma : & B & \to & B \\
 & x & \mapsto & x - \alpha \\
 & y & \mapsto & y
\end{array} \quad \textit{and} \quad \begin{array}{rlcl}
\delta : & B & \to & B \\
 & x & \mapsto & -f(x, y) \\
 & y & \mapsto & -g(x, y).
\end{array}$$

Then our algebra specified by the commutators in Equation 2.4 is isomorphic to $A[z; \sigma, \delta]$.

Proof. We first verify that σ is a ring homomorphism and δ is a σ-derivation.

$$\begin{aligned}
\sigma([x, y]) &= \sigma(x)\sigma(y) - \sigma(y)\sigma(x) = (x - \alpha)y - y(x - \alpha) = [x, y] - [\alpha, y] = y = \sigma(y) \\
\delta([x, y]) &= \delta(xy) - \delta(yx) = \sigma(x)\delta(y) + \delta(x)y - \sigma(y)\delta(x) - \delta(y)x \\
&= -(x - \alpha)g(x, y) - f(x, y)y + yf(x, y) + g(x, y)x \\
&= \alpha g(x, y) - [x, g(x, y)] - [f(x, y), y] = -g(x, y) = \delta(y)
\end{aligned}$$

where the last equality is given by the Jacobi identity (Equation 2.3).

We now show that the other two commutators are equivalent to Ore extension identity.

$$\begin{aligned}
zx &= xz - [x, z] = xz - \alpha z - f(x, y) = (x - \alpha)z - p(x, y) = \sigma(x)z + \delta(x) \\
zy &= yz - [y, z] = yz - g(x, y) = \sigma(y)z + \delta(y)
\end{aligned}$$

Thus our algebra $A = B[z; \sigma, \delta]$. □

Therefore, our algebra $A = k[x][y; \tau][z; \sigma, \delta]$, which is also an iterated Ore extension.

2.4 Case III: $[x, y] = z$, with y^2 in $[x, z]$

In this case, since the coefficient of y^2 in $[x, z]$ is nonzero, we know that $2b = \deg y^2 \leq \deg([x, z]) \leq a + c - 1 \leq 2a + b - 2$, which gives us $b \leq 2a - 2$. From here, we can deduce the following three inequalities regarding degrees:

$$\begin{aligned} c = \deg z = \deg[x, y] &\leq a + b - 1 \leq 3a - 3; \\ \deg[x, z] &\leq a + c - 1 \leq 4a - 4; \quad \text{and} \\ \deg[y, z] &\leq b + c - 1 \leq 5a - 6 \end{aligned}$$

In particular, $[x, z]$ is a linear combination of monomials from

$$\{1, x, x^2, x^3, y, xy, y^2, z\},$$

and $[y, z]$ is a linear combination of monomials from

$$\{1, x, x^2, x^3, x^4, y, xy, x^2y, y^2, z, xz\}.$$

We thus write

$$\begin{aligned} &[x, z] = \alpha z + (\beta_1 x + \beta_0)y + (\gamma_3 x^3 + \gamma_2 x^2 + \gamma_1 x + \gamma_0) + \delta y^2 \quad \text{and} \\ &[y, z] = (\rho_1 x + \rho_0)z + (\sigma_2 x^2 + \sigma_1 x + \sigma_0)y + (\tau_4 x^4 + \tau_3 x^3 + \tau_2 x^2 + \tau_1 x + \tau_0) + \varepsilon y^2. \end{aligned}$$

2.4.1 Characteristic not 2 or 3

Since $\delta \neq 0$ by assumption, notice that we may substitute y in place of $\delta y - \frac{\varepsilon}{\delta}x + \frac{\beta_0}{2}$, and z in place of δz when K has characteristic $p \neq 2$, which allows us to assume, without any loss of generality, that $\delta = 1$ and $\beta_0 = \varepsilon = 0$. Thus, we consider the algebra governed by

$$\begin{aligned} &[x, y] = z, \\ &[x, z] = \alpha z + \beta xy + (\gamma_3 x^3 + \gamma_2 x^2 + \gamma_1 x + \gamma_0) + y^2, \quad \text{and} \\ &[y, z] = (\rho_1 x + \rho_0)z + (\sigma_2 x^2 + \sigma_1 x + \sigma_0)y + (\tau_4 x^4 + \tau_3 x^3 + \tau_2 x^2 + \tau_1 x + \tau_0). \end{aligned}$$

The Jacobi identity puts a relation on these three commutators, and the coefficient of each monomial should vanish separately. This process yields the following 14 constraints.

$$
\begin{aligned}
yz &: 0 = \beta + 2\gamma_3 \\
x^2 z &: 0 = 3\gamma_3 + \sigma_2 \\
xz &: 0 = -3\alpha\gamma_3 + \beta\gamma_3 - \beta\rho_1 - \gamma_3\rho_1 + 2\gamma_2 + \sigma_1 \\
z &: 0 = \alpha^2\gamma_3 - \alpha\gamma_2 - \beta\rho_0 - \gamma_3\rho_0 + \gamma_1 + \sigma_0 \\
xy^2 &: 0 = \rho_1 - 3\gamma_3 \\
y^2 &: 0 = \alpha\gamma_3 - \gamma_2 + \rho_0 \\
x^2 y &: 0 = -\alpha\sigma_2 - 3\beta\gamma_3 + \beta\rho_1 - \beta\sigma_2 - \gamma_3\sigma_2 \\
xy &: 0 = \alpha\beta\gamma_3 - \alpha\sigma_1 - \beta\gamma_2 + \beta\rho_0 - \beta\sigma_1 - \gamma_3\sigma_1 \\
y &: 0 = -(\alpha + \beta + \gamma_3)\sigma_0 \\
x^4 &: 0 = -\alpha\tau_4 - \beta\tau_4 - 3\gamma_3^2 - \gamma_3\tau_4 + \gamma_3\rho_1 \\
x^3 &: 0 = \alpha\gamma_3^2 - \alpha\tau_3 - \beta\tau_3 - 4\gamma_3\gamma_2 - \gamma_3\tau_3 + \gamma_3\rho_0 + \gamma_2\rho_1 \\
x^2 &: 0 = \alpha\gamma_3\gamma_2 - \alpha\tau_2 - \beta\tau_2 - 3\gamma_3\gamma_1 - \gamma_3\tau_2 - \gamma_2^2 + \gamma_2\rho_0 + \gamma_1\rho_1 \\
x &: 0 = \alpha\gamma_3\gamma_1 - \alpha\tau_1 - \beta\tau_1 - 3\gamma_3\gamma_0 - \gamma_3\tau_1 - \gamma_2\gamma_1 + \gamma_1\rho_0 + \gamma_0\rho_1 \\
1 &: 0 = \alpha\gamma_3\gamma_0 - \alpha\tau_0 - \beta\tau_0 - \gamma_3\tau_0 - \gamma_2\gamma_0 + \gamma_0\rho_0
\end{aligned}
$$

We can immediately see the first 6 constraints give us easy substitutions.

$$
\begin{aligned}
yz &: \beta = -2\gamma_3 \\
x^2 z &: \sigma_2 = -3\gamma_3 \\
xy^2 &: \rho_1 = 3\gamma_3 \\
y^2 &: \rho_0 = \gamma_2 - \alpha\gamma_3 \\
xz &: \sigma_1 = 3\alpha\gamma_3 - \beta\gamma_3 + \beta\rho_1 + \gamma_3\rho_1 - 2\gamma_2 = 3\alpha\gamma_3 - \gamma_3^2 - 2\gamma_2 \\
z &: \sigma_0 = -\alpha^2\gamma_3 + \alpha\gamma_2 + \beta\rho_0 + \gamma_3\rho_0 - \gamma_1 = -\alpha^2\gamma_3 + \alpha\gamma_2 + \alpha\gamma_3^2 - \gamma_3\gamma_2 - \gamma_1
\end{aligned}
$$

This leaves us with the following 8 constraints.

$$\begin{aligned}
x^2y &: 0 = 3\gamma_3(\alpha - \gamma_3) \\
xy &: 0 = -3\alpha^2\gamma_3 + 4\alpha\gamma_3^2 - \gamma_3^3 - 2(\gamma_3 - \alpha)\gamma_2 \\
y &: 0 = \alpha^3\gamma_3 - 2\alpha^2\gamma_3^2 + \alpha\gamma_3^3 - \alpha^2\gamma_2 + 2\alpha\gamma_3\gamma_2 - \gamma_3^2\gamma_2 + \alpha\gamma_1 - \gamma_3\gamma_1 \\
x^4 &: 0 = \tau_4(\gamma_3 - \alpha) \\
x^3 &: 0 = \tau_3(\gamma_3 - \alpha) \\
x^2 &: 0 = \tau_2(\gamma_3 - \alpha) \\
x &: 0 = \tau_1(\gamma_3 - \alpha) \\
1 &: 0 = \tau_0(\gamma_3 - \alpha)
\end{aligned}$$

There are two families of solutions to this system of equations:

(a) either $\gamma_3 = \alpha$; or

(b) $\gamma_3 \neq \alpha$, which forces $\gamma_1 = \gamma_2 = \gamma_3 = \tau_0 = \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$.

In Case IIIa, our governing equations simplify to

$$\begin{aligned}
&[x, y] = z, \\
&[x, z] = \alpha z + y^2 - 2\alpha xy + (\alpha x^3 + \gamma_2 x^2 + \gamma_1 x + \gamma_0), \quad \text{and} \\
&[y, z] = (3\alpha x + \gamma_2 - \alpha^2)z + (-3\alpha x^2 + 2(\alpha^2 - \gamma_2)x - \gamma_1)y + f(x)
\end{aligned}$$

where $f \in K[x]$ is an arbitrary polynomial of degree at most 4.

In Case IIIb, our governing equations instead become

$$[x, y] = z, \quad [x, z] = \alpha z + y^2 + \gamma_0, \quad \text{and} \quad [y, z] = 0$$

where $\alpha \neq 0$.

2.4.2 Characteristic 3

Just as in the case of characteristic not 2 and 3, the same substitution can be made, which lead to the same set of constraints from the Jacobi identity, albeit with the coefficients reduced modulo 3.

$$
\begin{aligned}
yz &: 0 = \beta - \gamma_3 \\
x^2z &: 0 = \sigma_2 \\
xz &: 0 = \beta\gamma_3 - \beta\rho_1 - \gamma_3\rho_1 - \gamma_2 + \sigma_1 \\
z &: 0 = \alpha^2\gamma_3 - \alpha\gamma_2 - \beta\rho_0 - \gamma_3\rho_0 + \gamma_1 + \sigma_0 \\
xy^2 &: 0 = \rho_1 \\
y^2 &: 0 = \alpha\gamma_3 - \gamma_2 + \rho_0 \\
x^2y &: 0 = -\alpha\sigma_2 + \beta\rho_1 - \beta\sigma_2 - \gamma_3\sigma_2 \\
xy &: 0 = \alpha\beta\gamma_3 - \alpha\sigma_1 - \beta\gamma_2 + \beta\rho_0 - \beta\sigma_1 - \gamma_3\sigma_1 \\
y &: 0 = -(\alpha + \beta + \gamma_3)\sigma_0 \\
x^4 &: 0 = -\alpha\tau_4 - \beta\tau_4 - \gamma_3\tau_4 + \gamma_3\rho_1 \\
x^3 &: 0 = \alpha\gamma_3^2 - \alpha\tau_3 - \beta\tau_3 - \gamma_3\gamma_2 - \gamma_3\tau_3 + \gamma_3\rho_0 + \gamma_2\rho_1 \\
x^2 &: 0 = \alpha\gamma_3\gamma_2 - \alpha\tau_2 - \beta\tau_2 - \gamma_3\tau_2 - \gamma_2^2 + \gamma_2\rho_0 + \gamma_1\rho_1 \\
x &: 0 = \alpha\gamma_3\gamma_1 - \alpha\tau_1 - \beta\tau_1 - \gamma_3\tau_1 - \gamma_2\gamma_1 + \gamma_1\rho_0 + \gamma_0\rho_1 \\
1 &: 0 = \alpha\gamma_3\gamma_0 - \alpha\tau_0 - \beta\tau_0 - \gamma_3\tau_0 - \gamma_2\gamma_0 + \gamma_0\rho_0
\end{aligned}
$$

We can immediately see the first 6 constraints give us easy substitutions.

$$
\begin{aligned}
yz &: \beta = \gamma_3 \\
x^2z &: \sigma_2 = 0 \\
xy^2 &: \rho_1 = 0 \\
y^2 &: \rho_0 = \gamma_2 - \alpha\gamma_3 \\
xz &: \sigma_1 = -\beta\gamma_3 + \beta\rho_1 + \gamma_3\rho_1 + \gamma_2 = -\gamma_3^2 + \gamma_2 \\
z &: \sigma_0 = -\alpha^2\gamma_3 + \alpha\gamma_2 + \beta\rho_0 + \gamma_3\rho_0 - \gamma_1 = -\alpha^2\gamma_3 + \alpha\gamma_2 + \alpha\gamma_3^2 - \gamma_3\gamma_2 - \gamma_1
\end{aligned}
$$

This leaves us with the following 7 constraints.

$$
\begin{aligned}
xy &: 0 = (\gamma_3 - \alpha)(\gamma_2 - \gamma_3^2) \\
y &: 0 = (\gamma_3 - \alpha)((\gamma_3 - \alpha)(\alpha\gamma_3 - \gamma_2) - \gamma_1) \\
x^4 &: 0 = \tau_4(\gamma_3 - \alpha) \\
x^3 &: 0 = \tau_3(\gamma_3 - \alpha) \\
x^2 &: 0 = \tau_2(\gamma_3 - \alpha) \\
x &: 0 = \tau_1(\gamma_3 - \alpha) \\
1 &: 0 = \tau_0(\gamma_3 - \alpha)
\end{aligned}
$$

Once again, there are two solutions to this system of equations:

(c) either $\gamma_3 = \alpha$; or

(d) $\gamma_3 \neq \alpha$, which forces $\gamma_2 = \gamma_3^2$, $\gamma_1 = -\gamma_3(\gamma_3 - \alpha)^2$, and $\tau_0 = \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$.

In Case IIIc, our governing equations simplify to

$$\begin{aligned}
&[x, y] = z, \\
&[x, z] = \alpha z + y^2 + \alpha xy + (\alpha x^3 + \gamma_2 x^2 + \gamma_1 x + \gamma_0), \quad \text{and} \\
&[y, z] = (\gamma_2 - \alpha^2)z + ((\gamma_2 - \alpha^2)x - \gamma_1)y + f(x)
\end{aligned}$$

where $f \in K[x]$ is an arbitrary polynomial of degree at most 4.

In Case IIId, our governing equation instead become

$$\begin{aligned}
&[x, y] = z, \\
&[x, z] = \alpha z + y^2 + \gamma_3 xy + (\gamma_3 x^3 + \gamma_3^2 x^2 - \gamma_3(\gamma_3 - \alpha)^2 x + \gamma_0), \quad \text{and} \\
&[y, z] = \gamma_3(\gamma_3 - \alpha)z
\end{aligned}$$

where $\gamma_3 \neq \alpha$.

2.4.3 Characteristic 2

In contrast to the other characteristics, the same substitution cannot be made, since it involves division by 2. However, we can substitute y in place of $\delta y - \frac{\varepsilon}{\delta}x$, and z in place of δz; this allows us to assume, without any loss of generality, that $\delta = 1$ and $\varepsilon = 0$ (unlike in the other characteristics, we do not have any constraints on β_0). Thus, we consider the algebra governed by

$$\begin{aligned}
&[x, y] = z, \\
&[x, z] = \alpha z + (\beta_1 x + \beta_0)y + (\gamma_3 x^3 + \gamma_2 x^2 + \gamma_1 x + \gamma_0) + y^2, \quad \text{and} \\
&[y, z] = (\rho_1 x + \rho_0)z + (\sigma_2 x^2 + \sigma_1 x + \sigma_0)y + (\tau_4 x^4 + \tau_3 x^3 + \tau_2 x^2 + \tau_1 x + \tau_0).
\end{aligned}$$

The Jacobi identity puts a relation on these three commutators, which when split into monomials, each coefficient should vanish independently. This process yields the following constraints.

$$
\begin{aligned}
yz &: 0 = \beta_1 \\
x^2z &: 0 = \gamma_3 + \sigma_2 \\
xz &: 0 = \alpha\gamma_3 + \beta_1\gamma_3 + \beta_1\rho_1 + \gamma_3\rho_1 + \sigma_1 \\
z &: 0 = \alpha^2\gamma_3 + \alpha\gamma_2 + \beta_0\gamma_3 + \beta_1\rho_0 + \gamma_3\rho_0 + \gamma_1 + \sigma_0 \\
xy^2 &: 0 = \gamma_3 + \rho_1 \\
y^2 &: 0 = \alpha\gamma_3 + \gamma_2 + \rho_0 \\
x^2y &: 0 = \alpha\sigma_2 + \beta_1\gamma_3 + \beta_1\rho_1 + \beta_1\sigma_2 + \gamma_3\sigma_2 \\
xy &: 0 = \alpha\beta_1\gamma_3 + \alpha\sigma_1 + \beta_0\gamma_3 + \beta_0\rho_1 + \beta_1\gamma_2 + \beta_1\rho_0 + \beta_1\sigma_1 + \gamma_3\sigma_1 \\
y &: 0 = \alpha\beta_0\gamma_3 + \alpha\sigma_0 + \beta_0\gamma_2 + \beta_0\rho_0 + \beta_1\sigma_0 + \gamma_3\sigma_0 \\
x^4 &: 0 = (\alpha + \beta_1 + \gamma_3)\tau_4 + \gamma_3^2 + \gamma_3\rho_1 \\
x^3 &: 0 = (\alpha + \beta_1 + \gamma_3)\tau_3 + \alpha\gamma_3^2 + \gamma_2\rho_1 + \gamma_3\rho_0 \\
x^2 &: 0 = (\alpha + \beta_1 + \gamma_3)\tau_2 + \alpha\gamma_2\gamma_3 + \gamma_1\gamma_3 + \gamma_1\rho_1 + \gamma_2^2 + \gamma_2\rho_0 \\
x &: 0 = (\alpha + \beta_1 + \gamma_3)\tau_1 + \alpha\gamma_1\gamma_3 + \gamma_0\gamma_3 + \gamma_0\rho_1 + \gamma_1\gamma_2 + \gamma_1\rho_0 \\
1 &: 0 = (\alpha + \beta_1 + \gamma_3)\tau_0 + \alpha\gamma_0\gamma_3 + \gamma_0\gamma_2 + \gamma_0\rho_0
\end{aligned}
$$

The first 6 constraints give us easy substitutions.

$$
\begin{aligned}
yz &: \beta_1 = 0 \\
x^2z &: \sigma_2 = \gamma_3 \\
xy^2 &: \rho_1 = \gamma_3 \\
y^2 &: \rho_0 = \alpha\gamma_3 + \gamma_2 \\
xz &: \sigma_1 = \alpha\gamma_3 + \beta_1\gamma_3 + \beta_1\rho_1 + \gamma_3\rho_1 = \alpha\gamma_3 + \gamma_3^2 \\
z &: \sigma_0 = \alpha^2\gamma_3 + \alpha\gamma_2 + \beta_0\gamma_3 + \beta_1\rho_0 + \gamma_3\rho_0 + \gamma_1 = \alpha^2\gamma_3 + \alpha\gamma_3^2 + \alpha\gamma_2 + \gamma_2\gamma_3 + \beta_0\gamma_3 + \gamma_1
\end{aligned}
$$

The remaining 8 constraints thus simplify and become

$$\begin{aligned}
x^2y &: 0 = (\alpha+\gamma_3)\gamma_3 \\
xy &: 0 = (\alpha^2+\gamma_3^2)\gamma_3 = (\alpha+\gamma_3)^2\gamma_3 \\
y &: 0 = \alpha\gamma_3(\alpha+\gamma_3)^2 + \beta_0\gamma_3(\alpha+\gamma_3) + (\alpha+\gamma_3)^2\gamma_2 + (\alpha+\gamma_3)\gamma_1 \\
x^4 &: 0 = (\alpha+\gamma_3)\tau_4 \\
x^3 &: 0 = (\alpha+\gamma_3)\tau_3 \\
x^2 &: 0 = (\alpha+\gamma_3)\tau_2 \\
x &: 0 = (\alpha+\gamma_3)\tau_1 \\
1 &: 0 = (\alpha+\gamma_3)\tau_0.
\end{aligned}$$

Just like before, there are two solutions to this system of equations:

(e) either $\gamma_3 = \alpha$; or

(f) $\gamma_3 \neq \alpha$, which forces $\gamma_3 = \tau_0 = \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$, and $\gamma_1 = \alpha\gamma_2$.

In Case IIIe, our governing equations are

$$\begin{aligned}
&[x,y] = z, \\
&[x,z] = \alpha z + y^2 + \beta_0 y + (\alpha x^3 + \gamma_2 x^2 + \gamma_1 x + \gamma_0), \quad \text{and} \\
&[y,z] = (\alpha x + \alpha^2 + \gamma_2)z + (\alpha x^2 + \alpha\beta_0 + \gamma_1)y + f(x)
\end{aligned}$$

where $f \in K[x]$ is an arbitrary polynomial of degree at most 4.

In Case IIIf, our governing equations instead become

$$[x,y] = z, \quad [x,z] = \alpha z + y^2 + \beta_0 y + (\gamma_2 x^2 + \alpha\gamma_2 x + \gamma_0), \quad \text{and} \quad [y,z] = \gamma_2 z$$

where $\alpha \neq 0$.

2.5 Case IV: $[x,y]=z$, vanishing y^2 in $[x,z]$

We turn our attention to this final case, where $[x,y]=z$ immediately imposes the condition $c \le a+b-1$ on degrees. This tells us that $[x,z]$ has degree at most $2a+b-1$, which informs us that we cannot have any monomials $x^iy^jz^k$ with $j \ge 2$ or $k \ge 1$ unless $(i,j,k) = (0,0,1)$, and that if $j = 1$, then $k = 0$ and $i = 0, 1$. Thus, we can write $[x, z] = \alpha z + (\beta_1 x + \beta_0)y + f(x)$, where $\alpha, \beta_0, \beta_1 \in K$ and $f \in K[x]$, since we have intentionally suppressed the y^2 term.

Similarly, $[y, z]$ has degree at most $b + c - 1 \leq a + 2b - 2$, so it can only involve monomials y^2 and monomials of the form $x^i, x^j y, x^k z$. Thus, we can write $[y, z] = g_0(x)z + \gamma y^2 + g_1(x)y + g_2(x)$, where $\gamma \in K$ and $g_0, g_1, g_2 \in K[x]$.

We shall classify this by the values of coefficients of y and z in $[x, z]$. The coefficient $\beta_1 x + \beta_0$ of y can either be:

(a) linear in x, with $\beta_1 \neq 0$;

(b) constant, with $\beta_1 = 0$, $\beta_0 \neq 0$, and $\alpha \neq 0$;

(c) constant, with $\beta_1 = 0$, $\beta_0 \neq 0$, and $\alpha = 0$;

(d) vanishing, with $\beta_0 = \beta_1 = 0$, and $\alpha \neq 0$; or

(e) vanishing, with $\beta_0 = \beta_1 = 0$, and $\alpha = 0$.

2.5.1 Case IVa: $\beta_1 \neq 0$

Characteristic not 2

Allow us to write $f(x) = (\beta_1 x + \beta_0)\tilde{f}(x) + \delta$. Making a change of variables

$$x' = -\frac{2}{\beta_1}\left(x + \frac{\beta_0}{\beta_1}\right), \quad y' = y + \tilde{f}(x), \quad \text{and} \quad z' = -\frac{2}{\beta_1}z,$$

we can assume that $\beta_1 = 1$, $\beta_0 = 0$, and that $f(x)$ is some constant δ, giving us

$$[x, y] = z, \quad [x, z] = \alpha z - 2xy + \delta, \quad \text{and} \quad [y, z] = g_0(x)z + \gamma y^2 + g_1(x)y + g_2(x).$$

The Jacobi identity constraint yields

$$\begin{aligned} 0 = \delta g_0(x) - (\alpha + \gamma - 2)g_2(x) - 2x g_0(x)y - (\alpha + \gamma - 2)g_1(x)y \\ - (\gamma - 2)g_0(x)z + g_1(x)z - (\alpha + \gamma - 2)\gamma y^2 + 2(\gamma - 1)yz. \end{aligned}$$

From comparing the coefficients of y^2 and yz, we have that $\alpha = \gamma = 1$, which simplifies our Jacobi identity to

$$0 = \delta g_0(x) - 2x g_0(x)y + (g_0(x) + g_1(x))z.$$

Once again comparing coefficients, we get that $2xg_0(x) = 0$, which gives us $g_0(x) = 0$, and then $g_1(x) = -g_0(x) = 0$.

Thus, our commutators clean up to

$$[x, y] = z, \quad [x, z] = z - 2xy + \delta, \quad \text{and} \quad [y, z] = y^2 + g_2(x)$$

where we have some arbitrary $\delta \in K$ and $g_2 \in K[x]$.

Characteristic 2

Once again, allow us to write down the polynomial long division $f(x) = (\beta_1 x + \beta_0)\tilde{f}(x) + \delta$. Making a change of variables

$$x' = x + \frac{\beta_1}{\beta_0} \quad \text{and} \quad y' = y + \tilde{f}(x),$$

we can assume that $\beta_0 = 0$ and $f(x)$ is some constant δ, giving us

$$[x, y] = z, \quad [x, z] = \alpha z + \beta xy + \delta, \quad \text{and} \quad [y, z] = g_0(x)z + \gamma y^2 + g_1(x)y + g_2(x)$$

where $\beta \neq 0$. The Jacobi identity evaluates to

$$\begin{aligned} 0 = \delta g_0(x) &+ (\alpha + \beta + \gamma)g_2(x) + \beta x g_0(x) y + (\alpha + \beta + \gamma)g_1(x)y \\ &+ (\beta + \gamma)g_0(x)z + g_1(x)z - (\alpha + \beta + \gamma)\gamma y^2 + \beta yz. \end{aligned}$$

Isolating the coefficient of z from the equation above, we have $\beta = 0$, which contradicts our premises, leading to no new deformations.

2.5.2 Case IVb: $\beta_1 = 0$, $\beta_0 \neq 0$, $\alpha \neq 0$

This time, we can make a change of variables

$$x' = \frac{x}{\alpha} \quad \text{and} \quad y' = \alpha y + \frac{\alpha^2}{\beta_0} f(x),$$

which allows us to assume that $\alpha = 1$ and $f(x) = 0$, giving us

$$[x, y] = z, \quad [x, z] = z + \beta y, \quad \text{and} \quad [y, z] = g_0(x)z + \gamma y^2 + g_1(x)y + g_2(x)$$

where $\beta \neq 0$. The Jacobi identity evaluates to

$$\begin{aligned} 0 = -(\gamma + 1)g_2(x) &+ (\beta g_0(x) - (\gamma + 1)g_1(x))y \\ &+ (g_1(x) - \gamma g_0(x))z - \gamma(\gamma + 1)y^2 + 2\gamma yz. \end{aligned} \tag{2.5}$$

Characteristic not 2

When K has characteristic not 2, from comparing the coefficients of yz, we get $\gamma = 0$, which allows us to simplify the Jacobi identity to

$$0 = -g_2(x) + (\beta g_0(x) - g_1(x))y + g_1(x)z.$$

Once again comparing coefficients, we get that $g_0(x) = g_1(x) = g_2(x) = 0$. This forces our algebra to be governed by

$$[x, y] = z, \quad [x, z] = z + \beta y, \quad \text{and} \quad [y, z] = 0.$$

Characteristic 2

In characteristic 2, the Jacobi identity (Equation 2.5) evaluates to

$$0 = (\gamma + 1)g_2(x) + (\beta g_0(x) + (\gamma + 1)g_1(x))y + (g_1(x) + \gamma g_0(x))z + \gamma(\gamma + 1)y^2.$$

From comparing coefficients of y^2, we have either $\gamma = 0$ or 1.

If $\gamma = 1$, then our Jacobi identity simplifies to

$$0 = \beta g_0(x)y + (g_0(x) + g_1(x))z.$$

Once again comparing coefficients, we get $g_0(x) = g_1(x) = 0$, which condenses our constraints down to

$$[x, y] = z, \quad [x, z] = z + \beta y, \quad \text{and} \quad [y, z] = y^2 + g_2(x),$$

where we have an arbitrary polynomial $g_2 \in K[x]$.

On the other hand, if $\gamma = 0$, then our Jacobi identity takes the form

$$0 = g_2(x) + (\beta g_0(x) + g_1(x))y + g_1(x)z.$$

Comparing coefficients tells us that $g_0(x) = g_1(x) = g_2(x) = 0$, which means out constraints boil down to

$$[x, y] = z, \quad [x, z] = z + \beta y, \quad \text{and} \quad [y, z] = 0.$$

2.5.3 Case IVc: $\beta_1 = 0$, $\beta_0 \neq 0$, $\alpha = 0$

In this case, we can make a change of variables $y' = y + f(x)/\beta_0$ which allows us to assume that $f(x) = 0$, giving us

$$[x, y] = z, \quad [x, z] = \beta y, \quad \text{and} \quad [y, z] = g_0(x)z + \gamma y^2 + g_1(x)y + g_2(x)$$

where $\beta \neq 0$. The Jacobi identity evaluates to

$$0 = -\gamma g_2(x) + (\beta g_0(x) - \gamma g_1(x))y + (g_1(x) - \gamma g_0(x))z - \gamma^2 y^2 + 2\gamma yz.$$

From comparing coefficients of y^2, we have $\gamma = 0$, which simplifies our Jacobi identity to

$$0 = \beta g_0(x)y + g_1(x)z.$$

Once again comparing coefficients, we have $g_0(x) = g_1(x) = 0$, which leaves us with an algebra governed by

$$[x, y] = z, \quad [x, z] = \beta y, \quad \text{and} \quad [y, z] = g_2(x)$$

where $g_2 \in K[x]$ is an arbitrary polynomial.

2.5.4 Case IVd: $\beta_0 = \beta_1 = 0$, $\alpha \neq 0$

We now make a change of variables

$$x' = \frac{x}{\alpha} \quad \text{and} \quad y' = \alpha y$$

such that without any loss of generality, we can assume that $\alpha = 1$. The commutators are thus

$$[x, y] = z, \quad [x, z] = z + f(x), \quad \text{and} \quad [y, z] = g_0(x)z + \gamma y^2 + g_1(x)y + g_2(x),$$

and the Jacobi identity evaluates to

$$\begin{aligned} 0 = {} & g_0(x)f(x) - (\gamma + 1)g_2(x) - (\gamma + 1)g_1(x)y \\ & - \gamma g_0(x)z + g_1(x)z + [f(x), y] - (\gamma + 1)\gamma y^2 + 2\gamma yz. \end{aligned} \tag{2.6}$$

Characteristic not 2

Notice that yz does not occur in the expression of $[f(x), y]$, so comparing coefficients of yz, we get $\gamma = 0$. The Jacobi identity thus simplifies to

$$\mathrm{ad}_y(f(x)) = (g_0(x)f(x) - g_2(x)) - g_1(x)y + g_1(x)z.$$

We now explicitly compute $\mathrm{ad}_y(f(x))$, which requires us to first compute $\mathrm{ad}_x^n(y)$ for integers $n \geq 0$. In an ordered K-basis $\{z, y, f(x)\}$, ad_x behaves as the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which has eigenvalues 0 (with multiplicity 2) and 1.

The Jordan canonical form of this matrix is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

and repeated applications of the adjoint operator ad_x yields

$$\begin{aligned} \mathrm{ad}_x^n(y) &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned} \tag{2.7}$$

for $n \geq 2$. Putting these back into the Jacobi identity and comparing coefficients, we get

$$\begin{aligned} g_1(x)z &= \sum_{n=1}^{\infty} (-1)^n f^{(n)}(x)z = (f(x-1) - f(x))z, \\ -g_1(x)y &= 0, \quad \text{and} \\ g_0(x)f(x) - g_2(x) &= \sum_{n=2}^{\infty} (-1)^n f^{(n)}(x) = \big(f(x-1) - f(x) + f^{(1)}(x)\big)f(x). \end{aligned}$$

The first two constraints informs us that $g_1(x) = 0$, and $f(x)$ is periodic, satisfying the identity $f(x-1) = f(x)$, which forces $f(x) = 0$ when K has characteristic 0; alternatively, by Proposition 2.7, $f(x)$ takes on the form $\tilde{f}(x^p - x)$ for some $\tilde{f} \in K[x]$ when K has characteristic $p > 0$. Finally, the last condition simplifies to

$$g_0(x)\tilde{f}(x^p - x) - g_2(x) = -\tilde{f}'(x^p - x)\tilde{f}(x^p - x)$$

and our algebra is thus governed by the equations

$$[x, y] = z, \quad [x, z] = z + \tilde{f}(x^p - x), \quad \text{and} \quad [y, z] = g_0(x)z + (g_0(x) + \tilde{f}'(x^p - x))\tilde{f}(x^p - x)$$

when K has characteristic $p > 0$; conversely, our algebra takes the form

$$[x, y] = z, \quad [x, z] = z, \quad \text{and} \quad [y, z] = g_0(x)z$$

when K has characteristic 0.

Characteristic 2

In characteristic 2, the Jacobi identity (Equation 2.6) becomes

$$0 = g_0(x)f(x) + (\gamma+1)g_2(x) + (\gamma+1)g_1(x)y + \gamma g_0(x)z + g_1(x)z + [f(x), y] + (\gamma+1)\gamma y^2.$$

Comparing coefficients of y^2 informs us that either $\gamma = 0$ or 1. In any case, we recall from our earlier computation in Equation 2.7 that

$$\mathrm{ad}_y(f(x)) = (f(x+1) + f(x))z + \big(f(x+1) + f(x) + f^{(1)}(x)\big)f(x).$$

If $\gamma = 0$, then our Jacobi identity simplifies to

$$\mathrm{ad}_y(f(x)) = (g_0(x)f(x) + g_2(x)) + g_1(x)y + g_1(x)z,$$

which after grouping by monomials, gives us

$$\begin{aligned} g_1(x)z &= (f(x+1) + f(x))z, \\ g_1(x)y &= 0, \quad \text{and} \\ g_0(x)f(x) + g_2(x) &= \big(f(x+1) + f(x) + f^{(1)}(x)\big)f(x). \end{aligned}$$

Using Proposition 2.7, we obtain that $f(x) = \tilde{f}(x^2 - x)$, $g_1(x) = 0$, and $g_2(x) = (g_0(x) + \tilde{f}'(x^2 - x))\tilde{f}(x^2 - x)$, so our governing equations become

$$[x, y] = z, \quad [x, z] = z + \tilde{f}(x^2 - x), \quad \text{and} \quad [y, z] = g_0(x)z + (g_0(x) + \tilde{f}'(x^2 - x))\tilde{f}(x^2 - x)$$

where $\tilde{f}, g_0 \in K[x]$ are arbitrary polynomials.

On the other hand, if $\gamma = 1$, then our Jacobi identity instead becomes

$$\mathrm{ad}_y(f(x)) = g_0(x)f(x) + (g_0(x) + g_1(x))z,$$

which after grouping by monomials, gives us

$$\begin{aligned}(g_0(x) + g_1(x))z &= (f(x+1) + f(x))z \quad \text{and} \\ g_0(x)f(x) &= \big(f(x+1) + f(x) + f^{(1)}(x)\big)f(x).\end{aligned}$$

If $f(x) = 0$, then we get that $g_0(x) = g_1(x)$, and our governing equations simplify to

$$[x, y] = z, \quad [x, z] = z, \quad \text{and} \quad [y, z] = g_0(x)z + y^2 + g_0(x)y + g_2(x)$$

where $g_0, g_2 \in K[x]$ are arbitrary polynomials. Otherwise, if $f(x) \neq 0$, then $g_0(x) = f(x+1) + f(x) + f^{(1)}(x)$ and $g_1(x) = f^{(1)}(x)$, and our algebra is thus determined by the commutators

$$\begin{aligned}&[x, y] = z, \\ &[x, z] = z + f(x), \quad \text{and} \\ &[y, z] = (f(x+1) + f(x) + f^{(1)}(x))z + y^2 + f^{(1)}(x)y + g_2(x)\end{aligned}$$

where $f, g_2 \in K[x]$ are arbitrary polynomials.

2.5.5 Case IVe: $\beta_0 = \beta_1 = 0$, $\alpha = 0$

For this final case, we can treat all characteristics simultaneously. Our standing assumptions are

$$[x, y] = z, \quad [x, z] = f(x), \quad \text{and} \quad [y, z] = g_0(x)z + \gamma y^2 + g_1(x)y + g_2(x),$$

and the Jacobi identity is similar to that of Case IVc, being

$$0 = g_0(x)f(x) - \gamma g_2(x) - \gamma g_1(x)y - \gamma g_0(x)z + g_1(x)z + [f(x), y] - \gamma^2 y^2 + 2\gamma yz.$$

Equating the coefficient of y^2 on both sides yields $\gamma = 0$, which simplifies our Jacobi identity to

$$\mathrm{ad}_y(f(x)) = g_0(x)f(x) + g_1(x)z.$$

In an ordered K-basis $\{z, y, f(x)\}$, ad_x behaves as the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which has a single eigenvalue 0 (with multiplicity 3). The Jordan canonical form of this matrix is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Notice that this matrix is nilpotent of degree 3, and we have

$$\mathrm{ad}_x(y) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = z \quad \text{and} \quad \mathrm{ad}_x^2(y) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = f(x). \tag{2.8}$$

Substituting this into the Jacobi identity and comparing coefficients, we get

$$g_1(x)z = -f^{(1)}(x)z \quad \text{and} \quad g_0(x)f(x) = f^{(2)}(x)f(x).$$

When $f(x) = 0$, we have $g_0(x) = 0$, so our algebra is governed by

$$[x, y] = z, \quad [x, z] = 0, \quad \text{and} \quad [y, z] = g_1(x)y + g_2(x)$$

where $g_1, g_2 \in K[x]$ are arbitrary polynomials. On the other hand, if $f(x) \neq 0$, then we have $g_0(x) = f^{(2)}(x)$ and $g_1(x) = -f^{(1)}(x)$, which gives us

$$[x, y] = z, \quad [x, z] = f(x), \quad \text{and} \quad [y, z] = f^{(2)}(x)z - f^{(1)}(x)y + g_2(x)$$

where $g_2 \in K[x]$ is an arbitrary polynomial.

2.6 Reverse implication

In the above, we have shown that every filtered deformation can only possibly be isomorphic to one of the families as outlined in Theorem 1.2. We now show the reverse implication, that every one of our families is indeed a filtered deformation.

It is straightforward to show, with some computational assistance from a machine, that each of our cases above is a deformation. We first use the code from Appendix A to convert $[[x, y], z], [[x, z], y], [x, [y, z]]$ into normal form as a linear combination of the monomials $x^i y^j z^k$, from which we can verify that the Jacobi identity holds. Then, we shall use the filtered property to assign degrees to our symbols x, y, z, making sure that the sum of the degrees of any two symbols is strictly greater than the degree of the commutator of said two symbols. Using a degree-lexicographical ordering on words in x, y, z with $x > y > z$, Bergman's diamond lemma [5] shows that xy, xz, yz are leading monomials of a Gröbner basis for the ideal generated by the relations, and thus our algebra is indeed a filtered deformation of a polynomial ring.

This completes our classification of filtered deformations of $K[x, y, z]$.

Chapter 3

Polynomial Identity

We can now proceed to prove Theorem 1.1, which we do so by exhausting the cases that we obtained from Theorem 1.2.

Definition 3.1 (Polynomial identity ring)**.** A *polynomial identity* in a ring R is a polynomial in a finite number of variables such that it evaluates to zero for any choice of elements substituted in place of the variables. A *polynomial identity ring* is a ring with at least one polynomial identity that has at least one of the coefficients being ± 1.

Example 3.2 (Commutative ring)**.** Trivially, for any commutative ring R, the polynomial in two variables $f(s,t) = st - ts$ always evaluates to zero, so R is a polynomial identity ring.

Example 3.3 (Amitsur-Levitzski)**.** [1] Let R be the ring of $n \times n$ matrices with entries in a commutative ring. Then it is equipped with a polynomial identity in $2n$ variables

$$\sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) t_{\sigma(1)} t_{\sigma(2)} \dots t_{\sigma(2n)},$$

where S_{2n} denotes the symmetric group of $2n$ symbols.

From here onwards we will say a ring is PI if it is a polynomial identity ring.

3.1 Iterated Ore extensions

One way of showing a ring is PI is the following result of Brown and Zhang regarding iterated Ore extensions in positive characteristic.

Theorem 3.4. [6, Theorem 3.3] *Suppose K has characteristic $p > 0$, and let R be an n-step iterated Ore extension*

$$R = K[x_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n].$$

For $i \in \{1, \ldots, n-1\}$, let $R_{(i)} = K\langle x_1, \ldots, x_i\rangle$. Then R is PI if and only if $\sigma_i\big|_{Z(R_{i-1})}$ has finite order for $i \in \{2, \ldots, n\}$.

Using this result, we immediately have that in Cases I and II, our algebra is PI.

3.2 Central elements

A second way that we can show a ring is PI is by showing the existence of central elements. In private communication, Bell provides a proof that extends [4], and shows that finding a single central element in our ring R is sufficient for R to be PI. We replicate his proof below.

Lemma 3.5. *Let F be an algebraically closed field, let A be an F-algebra equipped with an ascending filtration by finite-dimensional K-vector spaces whose associated graded ring is a finitely generated graded Noetherian domain, and let D be the quotient division algebra of A. Then all subfields K of D containing F are finitely generated as extensions of F.*

Proof. Let B be the associated graded ring of A, and let K be a subfield of D containing F. Then by an argument of de Jong appearing in [2, Theorem 5.1], $B \otimes_F K$ is Noetherian. In particular, since $B \otimes_F K$ is an associated graded ring of $A \otimes_F K$, $A \otimes_F K$ is also Noetherian. Since $D \otimes_F K$ is a localization of $A \otimes_F K$, $D \otimes_F K$ is Noetherian. Since D is free over K, $D \otimes_F K$ is a free left and right $K \otimes_F K$-module, and thus a faithful flatness argument gives that $K \otimes_F K$ is Noetherian. It follows from [13] that K is finitely generated over F. □

Proposition 3.6. *Let A be a filtered deformation of a trivariate polynomial ring over the field F. If A has a central element $c \in A \setminus F$, then A is PI.*

Proof. This proof follows the strategy from [4]. We let x, y, z be generators for A such that $\bar{x}, \bar{y}, \bar{z}$ generate the associated graded ring of A. Given $u \in A$, we let $\bar{u}$ denote its image in the associated graded ring. Then the image of c in the associated graded ring of A is some nonzero homogeneous polynomial $\bar{c} := h(\bar{x}, \bar{y}, \bar{z})$. We let a, b, c denote the degrees of x, y, z respectively. Then we can put a degree-lexicographic order on monomials in $\bar{x}, \bar{y}, \bar{z}$ by ordering first by total degree and then lexicographically, using an order $\bar{x} > \bar{y} > \bar{z}$.

Let $\bar{x}^\alpha \bar{y}^\beta \bar{z}^\gamma$ denote the monomial that is degree-lexicographically greatest that occurs in $\bar{c}$ with nonzero coefficient. After possibly permuting x, y, and z, we may assume that $\alpha > 0$, and that this monomial is still degree-lexicographically greatest.

By considering images in the associated graded ring and using the fact that $\bar{x}^\alpha \bar{y}^\beta \bar{z}^\gamma$ is the leading monomial of a Gröbner basis for the ideal $(\bar{c})$ in $F[\bar{x}, \bar{y}, \bar{z}]$, we see that $c^i y^j z^k x^s$ with $i, j, k \geq 0$ and $s < \alpha$ forms an F-basis for A. We now let K denote the subfield of $\mathrm{Frac}(A)$ generated by c and y. Then since $\bar{c} = h$ has monomials involving x, $\bar{c}$ and $\bar{y}$ are algebraically independent over F, and so the field K has transcendence degree 2 and KV_n is spanned by monomials $z^k x^s$ with $s < \alpha$ such that $k \deg(z) + s \deg(x) \leq n$. In particular, $\dim_K KV_n/KV_{n-1} = O(1)$ since our basis for KV_n can have at most α elements of degree exactly n. By telescoping, for fixed $r \geq 1$, we have $\dim_K KV_n/KV_{n-r} = O(1)$ as $n \to \infty$.

On the other hand, if we let Ω_n denote the K-span of elements u in A such that $[y, u] \in V_n$, then we claim that

$$\limsup_{n \to \infty} (\dim_K(\Omega_n) - \dim_K(V_n)) = \infty. \tag{3.1}$$

The reason for this is that if we let L be such that $\deg(y) < p^L$, then if we let $N > L$ and consider the natural numbers

$$a_N = p^N, \quad a_{N-1} = p^N - p^{N-1}, \quad \ldots, \quad a_L = p^N - p^{N-1} - \cdots - p^L, \tag{3.2}$$

then for $i \in \{L, \ldots, N\}$, we have $a_i = a'_i p^i$ for some integer a'_i, and thus

$$[y, z^{a_i}] = -\operatorname{ad}_{z^{a_i}}(y) = -\operatorname{ad}^{p^i}_{z^{a'_i}}(y).$$

Since each application of $\operatorname{ad}_{z^{a'_i}}$ raises the degree by at most $a'_i \deg(z) - 1$, we see that $[y, z^{a_i}]$ has degree at most

$$\deg(z) a_i - p^i + \deg(y) \leq \deg(z)(p^N - p^{N-1} - \cdots - 2p^i) \leq \deg(z) \left(p^N - \sum_{j=L-1}^{N-1} p^j \right), \tag{3.3}$$

and so

$$[y, z^{a_i}] \subseteq \Omega_{b(N)},$$

where $b(N) \coloneqq \deg(z)(p^N - p^{N-1} - \cdots - p^L - p^{L-1})$.

But now this establishes Equation 3.1, since the sum $Kz^{a_L} + \cdots + Kz^{a_N} + KV_{b(N)-\deg(y)+1}$ is direct and by construction contained in $\Omega_{b(N)}$, so $\Omega_{b(N)}$ has dimension at least

$$N - L + 1 + \dim_K(V_{M-\deg(y)+1}).$$

Since N is arbitrary and as $\dim_K(KV_M/V_{M-\deg(y)+1}) = O(1)$, we see that Equation 3.1 holds.

Now the proof of [4, Corollary 1.6] shows that the centralizer E of y in $\mathrm{Frac}(A)$ is a division ring of infinite dimension over K. By construction $K \subseteq Z(E)$. Let K' is a maximal subfield of E. If $[K' : K] < \infty$, then $[K' : Z(E)] < \infty$, and so by [9, Theorem 15.4], we have $[E : Z(E)] < \infty$. Thus, we obtain $[E : K] = [E : Z(E)] \cdot [Z(E) : K] < \infty$, which is false. Hence, we have $[K' : K] = \infty$. By Lemma 3.5, K' is finitely generated as an extension of F and hence as an extension of K; and since $[K' : K] = \infty$, K' must have transcendence degree at least one over K. Hence, the transcendence degree of K'/F is strictly greater than $2 = \operatorname{tr deg}_F(K)$, so K' has transcendence degree at least 3 over F. Now [3, Theorem 1.1] gives that A is PI. □

Finally, we observe that although our algebras have positive characteristic, it suffices to find a non-trivial centre in characteristic zero, as it should descend to a non-trivial centre in positive characteristic.

Lemma 3.7. *Suppose R is a finitely generated commutative $\mathbb{Z}$-algebra which is an integral domain, $F = \mathrm{Frac}(R)$ its field of fractions, and let $\theta \colon R \to K$ be a ring homomorphism from R to a field K, so that we can view K as an R-module. Let A be an associative R-algebra equipped with an ascending filtration*

$$R = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots, \quad \textit{where} \quad A = \bigcup_{N \geq 0} A_N,$$

such that

- *(i) each A_N is a finitely generated free R-module and each quotient A_N/A_{N-1} is a free R-module;*
- *(ii) $A_M A_N \subseteq A_{M+N}$ for all $M, N \geq 0$; and*
- *(iii) A is generated as an R-algebra by finitely many elements $g_1, \ldots, g_r$, all lying in A_d for some fixed d.*

If $A \otimes_R F$ has a central element not contained in $F \cdot 1$, then $A \otimes_R K$ has a central element not contained in $K \cdot 1$.

Proof. Since the g_i generate A as an R-algebra, an element $a \in A$ is central if and only if $[a, g_i] = 0$ for $i = 1, \ldots, r$; the same condition holds for $A \otimes_R K$, since the images $g_i \otimes 1$

generate $A \otimes_R K$ as a K-algebra. Now suppose $a \in A_N \setminus F \cdot 1$ is central. For $b \in A_N$, we have $[b, g_i] \in A_{N+d}$ by assumption (ii), so there is an R-linear map of finitely generated free R-modules

$$\begin{array}{rccl} \Phi: & A_N & \to & A_{N+d}^r \\ & b & \mapsto & ([b, g_1], \ldots, [b, g_r]). \end{array}$$

By construction, we have $Z(A) \cap A_N = \ker \Phi$.

Fix R-bases of A_N and of $(A_{N+d})^{\oplus r}$, and let q and p denote the sizes of the respective bases. We can then represent Φ as a $p \times q$ matrix Y with entries in R, and we regard it as a matrix in F. Let $\rho = \operatorname{rank}_F(Y)$. Since $1, a \in \ker(\Phi)$ and they are linearly independent over F, $\rho \leq q-2$. Hence, every $(q-1) \times (q-1)$ minor of Y vanishes. If we let $\theta(Y) \in M_{p\times q}(K)$ denote the matrix obtained by applying θ entry-wise, we then see that every $(q-1) \times (q-1)$ minor of $\theta(Y)$ vanishes and so $\theta(Y)$ has rank at most $q-2$ as a matrix over K. Consequently,

$$\dim_K \ker(\Phi \otimes \mathrm{Id}_K) \geq 2$$

and so we see that $A \otimes_R K$ necessarily has a non-scalar central element in $A_N \otimes_R K$. $\square$

3.3 Trivial results

It now suffices to show that each subcase in Cases III and IV are either iterated Ore extensions or have a central element.

3.3.1 Case III

Cases IIIa and IIIc

In Cases IIIa and IIIc, writing its governing commutators as

$$\begin{aligned} &[x, y] = z, \\ &[x, z] = \alpha z + y^2 - 2\alpha xy + \alpha x^3 + \beta x^2 + \gamma x + \delta, \quad \text{and} \\ &[y, z] = (3\alpha x + \beta - \alpha^2)z - (3\alpha x^2 - 2(\alpha^2 - \beta)x + \gamma)y + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \end{aligned}$$

a central element for the algebra is

$$\begin{aligned}\Theta = {} & 30z^2 - 20y^3 + 10(3\alpha - 2a_4)yz + 30(3\alpha - 2a_4)x^2z - 20(3\alpha - 2a_4)xy^2 \\ & - 20(3\alpha - 2a_4)x^3y - 30(5\alpha^2 - 3a_4\alpha - 2\beta + a_3)xz + 10(4\alpha^2 - a_4\alpha - 4\beta + a_3)y^2 \\ & + 10(18\alpha^2 - 11a_4\alpha - 6\beta + 3a_3)x^2y + 10(3\alpha^3 - 2a_4\alpha^2 - 3\alpha\beta + a_4\beta + a_3\alpha - a2 + 3\gamma)z \\ & - 20(3\alpha^3 - 2a_4\alpha^2 - 3\alpha\beta + a_4\beta + a_3\alpha - a_2 + 3\gamma)xy + 10(a_1 - a_4\gamma - 6\delta)y + 12a_4x^5 \\ & - (3a_4\alpha - 8a_4^2 - 15a_3)x^4 - (21a_4\alpha^2 + 35a_3\alpha - 56a_4\beta - 8a_3a_4 - 20a_2)x^3 \\ & - (3a_4\alpha^3 + 5a_3\alpha^2 + 7a_4\alpha\beta + 40a_2\alpha - 15a_3\beta - 54a_4\gamma - 8a_2a_4 - 30a_1)x^2 \\ & - (9a_4\alpha\gamma + 30a_1\alpha - 15a_3\gamma - 48a_4\delta - 8a_1a_4 - 60a_0)x.\end{aligned}$$

This is computed using the code and method as outlined in Appendix A.

Case IIIb

In Case IIIb, with its governing commutators being

$$[x, y] = z, \quad [x, z] = \alpha z + y^2 + \beta, \quad \text{and} \quad [y, z] = 0,$$

our algebra is actually the differential polynomial ring $K[y, z][x, \mathrm{ad}_x]$.

Cases IIId and IIIf

Observe that when $[x, y] = z$ and $[y, z] = cz$ for some $c \in K$, so $\mathrm{ad}_y^n(x) = -c^{n-1}z$. Since $\mathrm{ad}_x(y^p) = -\mathrm{ad}_y^p(x)$ when K has characteristic $p > 0$, we should immediately get a central element

$$\Theta = y^p - c^{p-1}y.$$

More explicitly, in Case IIId, which is in characteristic 3, writing its governing commutators as

$$\begin{aligned}&[x, y] = z, \\ &[x, z] = \alpha z + y^2 + \beta xy + \beta x^3 + \beta^2x^2 - \beta(\beta - \alpha)^2x + \gamma, \quad \text{and} \\ &[y, z] = \beta(\beta - \alpha)z,\end{aligned}$$

a central element for the algebra is

$$\Theta = y^3 - (\alpha - \beta)^2\beta^2y.$$

In Case IIIf, which is in characteristic 2, writing its governing commutators as

$$[x,y] = z, \quad [x,z] = \alpha z + y^2 + \delta y + \beta x^2 + \alpha\beta x + \gamma, \quad \text{and} \quad [y,z] = \beta z,$$

a central element for the algebra is

$$\Theta = y^2 + \beta y.$$

Case IIIe

In Case IIIe, which is in characteristic 2, writing its governing commutators as

$$\begin{aligned}
&[x,y] = z,\\
&[x,z] = \alpha z + y^2 + \varepsilon y + \alpha x^3 + \beta x^2 + \gamma x + \delta, \quad \text{and}\\
&[y,z] = (\alpha x + \alpha^2 + \beta)z + (\alpha x^2 + \alpha\varepsilon + \gamma)y + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,
\end{aligned}$$

a central element for this algebra is

$$\Theta = x^4 + \alpha x^3 + (\alpha^2 + \beta + \varepsilon)x^2 + (\alpha\varepsilon + \gamma)x.$$

3.3.2 Case IV

Since Case IVa is a bit more involved, we shall defer it until the next section.

Case IVb

In Case IVb, one possibility of the governing set of commutators is

$$[x,y] = z, \quad [x,z] = z + \alpha y, \quad \text{and} \quad [y,z] = 0,$$

which is clearly the differential polynomial ring $K[y,z][x, \mathrm{ad}_x]$.

The other possible set of commutators, which happens only over characteristic 2, is

$$[x,y] = z, \quad [x,z] = z + \alpha y, \quad \text{and} \quad [y,z] = y^2 + f(x)$$

for some arbitrary polynomial $f(x)$, and a central element for the algebra is

$$\Theta = x^4 + (\alpha + 1)x^2 + \alpha x.$$

Case IVc

In Case IVc, writing its governing commutators as

$$[x,y]=z, \quad [x,z]=\alpha y, \quad \text{and} \quad [y,z]=f(x)$$

for some arbitrary polynomial $f(x)$, a central element for the algebra is

$$\Theta = x^{p^2} - \alpha^{p(p-1)/2} x^p$$

in characteristic $p>0$.

For characteristic 2, we can simply verify this by hand. For characteristic $p \geq 3$, notice that p is odd, so we can write $p = 2m+1$ for some integer m; moreover, as the odd powers of the adjoint operator obeys $\mathrm{ad}_x^{2n+1}(y) = \alpha^n z$, using the fact that p divides both $\binom{p^2}{k}$ and $\binom{p}{\ell}$ for $k \neq 0, p^2$ and $\ell \neq 0, p$ respectively, we can compute that

$$\mathrm{ad}_y(\Theta) = (-1)^{p^2} \mathrm{ad}_x^{p^2}(y) - (-1)^p \alpha^{p(p-1)/2} \mathrm{ad}_x^p(y) = \alpha^{p(p-1)/2}\alpha^m z - \alpha^{2m^2+2m} z = 0$$

as desired.

Case IVd

In Case IVd, one possibility for the governing set of commutators is

$$[x,y]=z, \quad [x,z]=z+f(x^p-x), \quad \text{and} \quad [y,z]=g(x)z+(g(x)+f^{(1)}(x^p-x))f(x^p-x),$$

which is the iterated Ore extension $K[x][z;\sigma,\varepsilon][y,\delta]$, where $\sigma, \varepsilon, \delta$ are defined as

$$\sigma: x \mapsto x-1, \quad \varepsilon: x \mapsto -f(x^p-x), \quad \text{and} \quad \delta: \begin{cases} x \mapsto z \\ z \mapsto (g(x)+f^{(1)}(x^p-x))f(x^p-x). \end{cases}$$

Another possible set of governing commutators is

$$[x,y]=[x,z]=z \quad \text{and} \quad [y,z]=f(x)z+y^2+f(x)y+g(x)$$

over characteristic 2, which has a central element

$$\Theta = x^2 + x$$

using the exact same logic as Cases IIId and IIIf from above.

The last possible set of governing commutators in this case is

$$\begin{aligned}
&[x, y] = z, \\
&[x, z] = z + f(x), \quad \text{and} \\
&[y, z] = (f(x+1) + f(x) + f^{(1)}(x))z + y^2 + f^{(1)}(x)y + g(x),
\end{aligned}$$

again over characteristic 2. This has a central element

$$\Theta = x^4 + x^2,$$

since we have

$$\mathrm{ad}_y(\Theta) = \mathrm{ad}_x^4(y) + \mathrm{ad}_x^2(y) = 2(z + f(x)) = 0.$$

Case IVe

In Case IVe, one possible set of governing commutators is

$$[x, y] = z, \quad [x, z] = 0, \quad \text{and} \quad [y, z] = f(x)y + g(x),$$

which is the skew polynomial ring $K[x, z][y; \sigma, \delta]$, where σ and δ are defined by

$$\sigma : \begin{cases} x \mapsto x \\ z \mapsto z + f(x) \end{cases} \qquad \text{and} \quad \delta : \begin{cases} x \mapsto -z \\ z \mapsto g(x) \end{cases}$$

respectively.

Finally, the other possible set of governing commutators is

$$[x, y] = z, \quad [x, z] = f(x), \quad \text{and} \quad [y, z] = f^{(2)}(x)z - f^{(1)}(x)y + g(x)$$

which is the iterated Ore extension $K[x][z, \varepsilon][y; \sigma, \delta]$, where $\varepsilon, \sigma, \delta$ are defined by

$$\varepsilon : x \mapsto -f(x), \quad \sigma : \begin{cases} x \mapsto x \\ z \mapsto z - f^{(1)(x)}, \end{cases} \qquad \text{and} \quad \delta : \begin{cases} x \mapsto -z \\ z \mapsto f^{(2)}(x)z + g(x). \end{cases}$$

3.4 Central element for Case IVa

In Case IVa, we need to apply some more advanced techniques. Our governing commutators are

$$[x, y] = z, \quad [x, z] = z - 2xy + \alpha, \quad \text{and} \quad [y, z] = y^2 + f(x)$$

where f is an arbitrary polynomial. It is easy to verify that for

$$\tilde{\Theta} = z^2 - 2yz + 2xy^2 + 2y^2 - 2\alpha y + f(x) + xf(x),$$

we have $[x, \tilde{\Theta}] = 0$ and $[y, \tilde{\Theta}] = xyf(x) - f(x)yx$. Thus, it suffices to find a polynomial $g \in K[x]$ such that $\mathrm{ad}_y(g(x)) = f(x)yx - xyf(x)$, as $\Theta = \tilde{\Theta} + g(x)$ would be our desired central element. By linearity, it suffices to find $g_n \in K[x]$ such that $\mathrm{ad}_y(g_n(x)) = x^n yx - xyx^n$ for all non-negative integers n.

In private communication, Bell provides us with the following argument for showing the existence of $g_n(x)$. We first prove a lemma about the annihilator of y in the subalgebra generated by x.

Lemma 3.8. *Consider our algebra as a left $K[u, v]$-module, where u and v act as left and right multiplication by x respectively. Denote $I = \mathrm{Ann}(y)$, the annihilator of y in $K[u, v]$. Then we can find polynomials $g_n \in K[x]$ such that $g_n(u) - g_n(v) - uv^n + u^n v \in I$ for every non-negative integer n.*

Proof. First off, by computation, we see that

$$((u - v)^2 + u + v)y = x^2y - 2xyx + yx^2 + xy + yx = [x, z] + xy + yx = \alpha,$$

so we know that

$$(u - v)((u - v)^2 + u + v)y = [x, \alpha] = 0,$$

and thus $(u - v)((u - v)^2 + u + v) \in I$. Let J be the ideal generated by $(u - v)^2 + u + v$.

Since $u^n - v^n$ is divisible by $u - v$ for all $n \geq 0$, by linearity, we know that $g_n(u) - g_n(v)$ is divisible by $u - v$, and thus $g_n(u) - g_n(v) - uv^n + u^n v$ is also divisible by $u - v$. Now, observe that as $u - v$ and $(u - v)^2 + u + v$ are coprime to each other, it really suffices to show that we can find $g_n \in K[x]$ such that $g_n(u) - g_n(v) - uv^n + u^n v \in J$ for every non-negative integer $n \geq 0$.

As we do not need to concern ourselves with the case of characteristic 2, we can see that $(u - v)^2 + u + v = 0$ defines a curve[1] of genus 0. We can parametrize the curve as

$$u = -\frac{1}{2}(t^2 + t) \quad \text{and} \quad v = -\frac{1}{2}(t^2 - t),$$

[1] In fact, it is a parabola.

which gives us the K-algebra isomorphism $\Psi\colon K[u,v]/J \to K[t]$ given by the parametrization above. Notice that $K[t]$ has a K-algbera automorphism of order 2, namely $\sigma : t \mapsto -t$, which in turn induces a K-algebra automorphism $\tilde{\sigma}$ over $K[u,v]/I$ that interchanges u and v.

We now claim that $S = \{\Psi(q(u) - q(v)) : q(\theta) \in K[\theta]\}$ is the set of odd polynomials in $K[t]$. To see this, observe that $q(u) - q(v)$ is sent to its negative under $\tilde{\sigma}$, so every element of S is sent to its negative under σ and hence is odd. On the other hand, $\Psi(u^n - v^n)$ is a degree $2n-1$ polynomial with leading coefficient $2^{-n+1}n$, so S contains a basis for the odd polynomials in t. Thus, to obtain the result, it suffices to note that $uv^n - v^n u = uv(v^{n-1} - u^{n-1})$ is sent to its negative by $\tilde{\sigma}$ and hence has odd image in $K[t]$. This proves the claim. $\square$

Immediately from the lemma above, we have that

$$\begin{aligned} 0 &= (g_n(u) - g_n(v) - uv^n + u^n v)(y) = g_n(x)y - yg_n(x) - xyx^n + x^n yx \\ &= -\operatorname{ad}_y(g_n(x)) + x^n yx - xyx^n \end{aligned}$$

as desired.

We have thus exhausted our cases, and this concludes the proof of Theorem 1.1.

Chapter 4

Conclusion

Throughout this work, we successfully classified all possible filtered deformations of $K[x, y, z]$ (Theorem 1.2), and then prove that all such deformations satisfy a polynomial identity when K is algebraically closed and has positive characteristic (Theorem 1.1). In particular, this latter result verifies a special case of a question posed by Etingof in [8, Question 1.1].

A natural next place to explore would be the filtered deformations of a quadrivariate polynomial ring. The same process outlined here can be repeated for one more variable, although in that case, there will be a total of $\binom{4}{2} = 6$ commutators, and $\binom{4}{3} = 4$ constraints due to the Jacobi identity, which greatly increases the number of computations needed to complete the classification.

In the case of four symbols, we should once again have two ways to show that a K-algebra is PI. One way is showing that it is an iterated Ore extension, since Theorem 3.4 still holds; on the other hand, a similar argument to Proposition 3.6 would now require two algebraically independent central elements, since we would need to descend twice to get to the case of Krull dimension 2.

If we are unable to find a counterexample in the case of four symbols, then we should explore the possibility that Etingof's question indeed holds true for filtered deformations of polynomial rings. A potential way to prove that is via induction: we can hope that the case of n variables always simplifies down to the case of $n - 1$ variables. Nevertheless, these deformations currently still elude our understanding, and whether we can resolve Etingof's question for all polynomial rings remains unclear.

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Appendix A

Computing in our Monomial Basis

A.1 The ncalg module

As we often need to write polynomials as a linear combination over $x^i y^j z^k$, we have written an algorithm in Python that recursively uses the commutator between any two symbols to sort the symbols in each monomial until they are in normal form. In the process of doing so, we have written a full Python module that performs arithmetic on non-commutative polynomials, implemented on top of the symbolic algebra package SymPy. [10]

```
from __future__ import annotations
from dataclasses import dataclass
from typing import Any
from sympy import Add, Expr, Integer, Mul, Symbol, simplify

def commutator(a, b):
    poly_a = a.to_polynomial()
    poly_b = b.to_polynomial()
    return poly_a * poly_b - poly_b * poly_a
```

We first introduce a helper function `order_wdeg` that sorts the terms of a polynomial into descending order, where the monomials are compared by their weighted degree. Notably, the polynomial is treated as a list of monomials, where each monomial shall be entered as a tuple `(coefficient, (i,j,k))`, with i, j, k being the exponents of x, y, z respectively. This implements the same functionality as the function `sort/polynomial` over the order `wdeg` in Maple.

```
def order_wdeg(poly: list[tuple[Any, tuple[int, ...]]],
               wdegs: tuple[int, ...]):
    """Orders the terms of a polynomial using a weighted degree."""
```

```python
    total_deg = lambda monom : sum(l * r for l, r in zip(monom[1], wdegs))
    return sorted(poly, key = total_deg, reverse=True)
```

Before we can start sorting, we must first create custom data types for (non-commutative) symbols, monomials, and polynomials. These classes are equipped with methods for simple arithmetic, conversion into SymPy objects (allowing us to utilize its pretty printing and robust commutative algebra functionalities), and casting of data types from symbols to monomials to polynomials. The class `NCsym` represents a single non-commutative symbol as a string.

```python
@dataclass(frozen=True)
class NCsym:
    """Represents a single non-commutative symbol."""

    symbol: str

    def __mul__(self, other):
        return self.to_monomial() * other

    def __rmul__(self, other):
        return other * self.to_monomial()

    def __pow__(self, power):
        if not isinstance(power, int) or power < 0:
            raise ValueError("Power must be a non-negative integer")
        if power == 0:
            return NCmonomial(Integer(1), ())
        else:
            return NCmonomial(Integer(1), (self,) * power)

    def to_sympy(self, comm=False):
        return Symbol(self.symbol, commutative=comm)

    def __repr__(self):
        return self.to_sympy().__repr__()

    def to_monomial(self):
        return NCmonomial(Integer(1), (self,))

    def to_polynomial(self):
        return self.to_monomial().to_polynomial()
```

Similarly, the class `NCmonomial` represents a monomial, comprising of a commutative coefficient represented as a SymPy object, and an array of non-commutative symbols.

```python
@dataclass(frozen=True)
class NCmonomial:
    """Represents a single monomial involving a (commutative) coefficient
    and a string of non-commutative symbols."""

    coeff: Expr
```

```python
    base: tuple[NCsym, ...]

    def __add__(self, other):
        if isinstance(other, NCmonomial):
            if self.base == other.base:
                return NCmonomial(Add(self.coeff, other.coeff), self.base)
            else:
                return self.to_polynomial() + other
        elif isinstance(other, NCsym):
            if self.base == (other,):
                return NCmonomial(Add(self.coeff, Integer(1)), self.base)
            else:
                return self.to_polynomial() + other
        else:
            return TypeError(f"Unsupported addition, NCmonomial and {type(other)}")

    def __mul__(self, other):
        if isinstance(other, NCsym):
            return NCmonomial(self.coeff, self.base + (other,))
        elif isinstance(other, NCmonomial):
            return NCmonomial(Mul(self.coeff, other.coeff), self.base + other.base)
        elif isinstance(other, NCpolynomial):
            return NotImplemented
        return NCmonomial(Mul(self.coeff, other), self.base)

    def __rmul__(self, other):
        if isinstance(other, NCsym):
            return NCmonomial(self.coeff, (other,) + self.base)
        elif isinstance(other, NCpolynomial):
            return NotImplemented
        return NCmonomial(Mul(other, self.coeff), self.base)

    def __neg__(self):
        return NCmonomial(-self.coeff, self.base)

    def __sub__(self, other):
        if isinstance(other, NCsym):
            return self - other.to_monomial()
        return self + (-other)

    def to_sympy(self, commutative=False):
        return Mul(self.coeff, Mul(*(sym.to_sympy(commutative) for sym in self.base)))

    def __repr__(self):
        return self.to_sympy().__repr__()

    def to_polynomial(self):
        return NCpolynomial((self,))
```

However, unlike `NCsym`, `NCmonomial` is also equipped with several methods that handles the swapping of symbols. In particular, the three extra methods that we have helps decide whether the monomial is sorted, where does it need sorting, and computes the polynomial that we must carry once we apply the commutator and swap two adjacent symbols.

```python
    def is_sorted(self, order: tuple[NCsym, ...]) -> bool:
```

```python
        """Decides whether the current monomial has all its symbols in the
        given order."""

        order_dict = {sym: n+1 for n, sym in enumerate(order)}
        mapped = [order_dict.get(sym, 0) for sym in self.base]
        return all(mapped[i] <= mapped[i+1] for i in range(len(mapped) - 1))

    def first_unsorted_index(self, order: tuple[NCsym, ...]) -> int:
        """Returns, for the current monomial,the first index among its symbols
        that is not in order."""

        order_dict = {sym: n+1 for n, sym in enumerate(order)}
        mapped = [order_dict.get(sym, 0) for sym in self.base]
        for i in range(len(mapped) - 1):
            if mapped[i] > mapped[i+1]:
                return i
        return -1

    def swap(self, n: int, comm: NCpolynomial):
        """Performs a swap of two symbols, and also term that we have to
        subtract using the commutator."""

        if n < 0 or n >= len(self.base) - 1:
            raise IndexError("Index out of bounds for swapping")
        new_base = list(self.base)
        new_base[n], new_base[n + 1] = new_base[n + 1], new_base[n]
        swapped_monomial = NCmonomial(self.coeff, tuple(new_base))
        front = (NCmonomial(-self.coeff, self.base[:n]),)
        end = (NCmonomial(Integer(1), self.base[n+2:]),)
        commutator_term = NCpolynomial(front) * comm * NCpolynomial(end)
        return swapped_monomial, commutator_term
```

Finally, `NCpolynomial` represents a polynomial, which is stored as an array of monomials.

```python
@dataclass(frozen=True)
class NCpolynomial:
    """Represents a polynomial with non-commutative monomials."""

    monomials: tuple[NCmonomial, ...]

    def __add__(self, other):
        if isinstance(other, NCpolynomial):
            return NCpolynomial(self.monomials + other.monomials)
        elif isinstance(other, NCmonomial):
            return NCpolynomial(self.monomials + (other,))
        elif isinstance(other, NCsym):
            return NCpolynomial(self.monomials + (NCmonomial(Integer(1), (other,)),))
        return self + NCmonomial(other, ())

    def __mul__(self, other):
        if isinstance(other, NCpolynomial):
            return NCpolynomial(tuple(m1 * m2 for m1 in self.monomials for m2 in other.monomials))
        elif isinstance(other, NCmonomial):
            return NCpolynomial(tuple(m * other for m in self.monomials))
        elif isinstance(other, NCsym):
            return NCpolynomial(tuple(m * other for m in self.monomials))
```

```python
        return NCpolynomial(tuple(m * other for m in self.monomials))

    def __rmul__(self, other):
        return NCpolynomial(tuple(other * m for m in self.monomials))

    def __neg__(self):
        return NCpolynomial(tuple(-m for m in self.monomials))

    def __sub__(self, other):
        if isinstance(other, NCpolynomial):
            return self + (-other)
        elif isinstance(other, NCmonomial):
            return self + (-other)
        elif isinstance(other, NCsym):
            return self + NCpolynomial((NCmonomial(-Integer(1), (other,)),))
        else:
            raise TypeError(f"Unsupported subtraction, NCpolynomial and {type(other)}")

    def to_sympy(self, commutative=False):
        return Add(*(mono.to_sympy(commutative) for mono in self.monomials))

    def __repr__(self):
        return self.to_sympy().__repr__()

    def to_polynomial(self):
        return self
```

After all of the build-up, we are finally able to compose our `sort` function. This function keeps track of a to-do list of mononomials, performs a required swap of symbols on each monomial, and then adds both this new monomial and the commutator terms back into the to-do list; we shall recurse through this list until we are done.

```python
    def sort(self, order: tuple[NCsym, ...],
             comm: tuple[tuple[NCpolynomial, ...], ...]) -> NCpolynomial:
        """Using the commutators, sorts every monomial into the desired order."""

        stack = list(self.monomials)
        done = {}
        order_dict = {sym: n for n, sym in enumerate(order)}
        while stack:
            mono = stack.pop()
            index = mono.first_unsorted_index(order)
            if index == -1:
                done[mono.base] = done.get(mono.base, Integer(0)) + mono.coeff
            else:
                sym1 = mono.base[index]
                sym2 = mono.base[index + 1]
                val1 = order_dict.get(sym1, -1)
                val2 = order_dict.get(sym2, -1)
                commutator = None
                try:
                    commutator = comm[val2][val1] if val1 != -1 and val2 != -1 else NCpolynomial(())
                except IndexError:
                    commutator = NCpolynomial(())
                swapped_mono, commutator_term = mono.swap(index, commutator)
```

```
            stack.append(swapped_mono)
            stack.extend(commutator_term.monomials)
    # iterate over dictionary to return a sorted NCpolynomial
    monomials = []
    for base, coeff in done.items():
        if coeff != 0:
            monomials.append(NCmonomial(simplify(coeff), base))
    return NCpolynomial(tuple(monomials))
```

We are aware that our sorting algorithm is a worsened version of bubble sort, possibly with average time complexity $O(n^3)$. However, for polynomial of low degree over three non-commutative symbols, this algorithm works fine enough; although it will probably need improvement if we were to work over four or more symbols.

A.2 Finding central elements using ncalg

We demonstrate how we use this package to find a centre for our algebra, specifically in Cases III and IV. For the rest of the section, quantities inside angle brackets `quantity` shall be replaced by various expressions before the code is executed. Firstly, we initialize our symbols.

```
import ncalg as nc
from sympy import Poly, symbols

x = nc.NCsym("x")
y = nc.NCsym("y")
z = nc.NCsym("z")
a,b,c,d = symbols('alpha beta gamma delta')
a0, a1, a2, a3, a4 = symbols('a0 a1 a2 a3 a4')
r,s = symbols('r s')
```

We then input our three commutators into a tuple, which can then be retrieved later on by our sorting algorithm.

```
commutator = ((0, <insert [x,y]>, <insert [x,z]>),
              (0, 0, <insert [y,z]>))
```

We can then input our guess for a centre as `theta`, and the algorithm computes the commutator $[x, \Theta]$, before outputting the every term in descending order according to weighted degree.

```
theta =<insert potential centre>
result = nc.commutator(x,theta).sort((x,y,z), commutator).to_sympy(commutative=True)
p = Poly(result, symbols('x y z'))
for m in nc.order_wdeg(list(zip(p.coeffs(), p.monoms())), (1,3,4)):
    print(m)
```

Replacing `nc.commutator(x,theta)` with `nc.commutator(y,theta)` in the second line in the above code block instead computes $[y, \Theta]$. Once we find Θ such that $[x, \Theta] = [y, \Theta] = 0$, we are done, since $[z, \Theta] = [[x, y], \Theta]$ in Cases III and IV, and this vanishes due to the Jacobi identity.